

# Lecture #26

## \* Last time:

1) Any analytic f-n  $f(z)$  on the annulus  $r < |z - z_0| < R$  has a Laurent series

$$f(z) = \sum_{j=0}^{\infty} a_j (z - z_0)^j + \sum_{j=1}^{\infty} a_{-j} (z - z_0)^{-j}$$

with both sums converging in this annulus (converging uniformly in any subannulus  $p_1 \leq |z - z_0| \leq p_2 \quad \forall r < p_1 < p_2 < R$ )

[Remark: While we had explicit formulas for  $a_j, j = -\infty, \dots, +\infty$  in terms of integrals, in practice they are rarely used (as we shall see)]

2) Applying the above to  $r=0$ , we obtain three possibilities for an isolated singularity of  $f(z)$  at  $z_0$ :

a)  $z_0$  - removable singularity if  $a_{-j} = 0 \quad \forall j > 0$ .

b)  $z_0$  - pole of order  $m > 0$  if  $a_{-m} \neq 0, a_{-j} = 0 \quad \forall j > m$

c)  $z_0$  - essential singularity if there are  $\infty$  many  $j > 0$  s.t.  $a_{-j} \neq 0$

## Examples:

1)  $f(z) = \frac{z^2}{z}$  has a removable singularity at  $z_0 = 0$ , where it's not defined.

Indeed,  $f(z) = z$  on  $\mathbb{C} \setminus \{0\} \Rightarrow$  its Laurent series has  $a_j = \begin{cases} 1, & j=1 \\ 0, & j \neq 1 \end{cases}$ .

2)  $f(z) = \frac{\sin z}{z}$  has a removable singularity at  $z_0 = 0$ , where it's not defined

Indeed,  $\sin z$  has a Taylor series  $z - \frac{z^3}{3!} + \frac{z^5}{5!} - \dots$  whose radius of convergence is  $R = \infty \Rightarrow \sin z = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \dots \Rightarrow \frac{\sin z}{z} = 1 - \frac{z^2}{3!} + \frac{z^4}{5!} - \dots$

for any  $z \neq 0$  and the latter sum converges everywhere. Hence, Laurent series of  $f(z)$  at  $z_0 = 0$  is  $1 - \frac{z^2}{3!} + \frac{z^4}{5!} - \dots \Rightarrow a_j = 0 \quad \forall j > 0$

3)  $f(z) = \frac{\sin z}{z^3}$  has a pole of order 3 at  $z_0 = 0$  (arguing as above we get

Laurent series  $\frac{1}{z^3} - \frac{1}{3!} \cdot \frac{1}{z} + \frac{1}{5!} z - \dots$ )

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Ex 1: Describe zeroes & singularities of  $f(z) = \cos(\frac{1}{z})$ .

Zeroes

Recall that  $\cos w = 0 \iff w = \frac{\pi}{2} + \pi k, k \in \mathbb{Z}$ .

Hence, zeroes of  $f(z)$  are  $z = \frac{1}{\frac{\pi}{2} + \pi k} = \frac{2}{2k+1} \cdot \frac{1}{\pi}, k \in \mathbb{Z}$ .

We also have to describe order of each zero!

As  $f'(z) = -\sin(\frac{1}{z}) \cdot \frac{-1}{z^2} = \frac{\sin(\frac{1}{z})}{z^2}$  and  $\sin(\frac{1}{z}) \neq 0$  for above  $z \implies$  all of order 1

So:  $f(z)$  has order 1 zeroes at  $\{ \frac{2}{(2k+1)\pi} \mid k \text{ integer} \}$

Poles

Recall that  $\cos w = 1 - \frac{w^2}{2!} + \frac{w^4}{4!} - \frac{w^6}{6!} + \dots$  has  $R = \infty$  (radius of conv.)

$$\implies \cos \frac{1}{z} = 1 - \frac{1}{2!} \cdot z^{-2} + \frac{1}{4!} z^{-4} - \frac{1}{6!} z^{-6} + \dots$$

Hence, the above is a Laurent series of  $f(z)$  near the only isolated singularity  $z_0 = 0$  (where  $f$  is not defined)  $\implies z_0 = 0$  - essential singularity

Let's now discuss several structural results that allow to understand which of 3 types singularity we have by analyzing  $f(z)$  as  $z \rightarrow z_0$ .

Lemma 1: If  $f(z)$  has a removable singularity at  $z_0$ , then:

- i)  $f$ -bounded in  $D_\epsilon(z_0) \setminus \{z_0\}$  for some  $\epsilon > 0$
- ii)  $f$  has a finite limit as  $z \rightarrow z_0$
- iii)  $g(z) = \begin{cases} f(z), & z \in D_\epsilon(z_0) \setminus \{z_0\} \\ \lim_{z \rightarrow z_0} f(z), & z = z_0 \end{cases}$  is analytic on  $D_\epsilon(z_0)$

Indeed, by definition, Laurent series of  $f(z)$  near  $z_0$  is  $\sum_{j=0}^{\infty} a_j(z-z_0)^j$  which converges on  $D_\epsilon(z_0) \setminus \{z_0\}$ , but also at  $z_0$  where it equals  $a_0$ .

Hence, by Theorem 3 of lecture 22, this Laurent = power series sums to a function  $g(z)$  analytic on  $D_\epsilon(z_0)$ . All parts follow now

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As above lemma shows, removable singularities are not really of big interest.

Lemma 2: If  $f(z)$  has a pole of order  $m > 0$  at  $z_0$ , then:

- i)  $(z-z_0)^m f(z)$  has a removable singularity at  $z_0$
- ii)  $|(z-z_0)^l f(z)| \xrightarrow{z \rightarrow z_0} 0 \quad \forall 0 \leq l < m$
- iii)  $f(z) \rightarrow \infty$  as  $z \rightarrow z_0$

As  $f(z) = \frac{a_{-m}}{(z-z_0)^m} + \frac{a_{-m+1}}{(z-z_0)^{m-1}} + \dots$  converging in some  $D_\epsilon(z_0) \setminus \{z_0\}$ , we get

$(z-z_0)^m f(z) = a_{-m} + a_{-m+1}(z-z_0) + \dots$  converging in the same  $D_\epsilon(z_0) \setminus \{z_0\}$ ,  
but also at  $z = z_0$  (as we get  $a_{-m} + 0 + 0 + \dots$ )

Hence, as in the proof of Lemma 1, we see that  $(z-z_0)^m f(z)$  has a removable singularity and  $(z-z_0)^m f(z) \xrightarrow{z \rightarrow z_0} a_{-m} \neq 0$ . Hence  $\forall 0 \leq l < m$ :

$$|(z-z_0)^l f(z)| = \left| \frac{(z-z_0)^m f(z)}{(z-z_0)^{m-l}} \right| \xrightarrow{z \rightarrow z_0} \infty. \text{ For } l=0, \text{ this implies } f(z) \xrightarrow{z \rightarrow z_0} \infty$$

Def:  $z_0$  is called a simple pole of  $f(z)$  if it is a pole of order 1

Corollary 1:  $f(z)$  has a pole of order  $m$  at  $z_0 \iff$  in some  $D_\epsilon(z_0) \setminus \{z_0\}$ :  
 $f(z) = \frac{g(z)}{(z-z_0)^m}$  with  $g(z)$  analytic in  $D_\epsilon(z_0)$  and  $g(z_0) \neq 0$ .

$\Rightarrow$ : follows from above proof of Lemma 2

$\Leftarrow$ : if  $g(z)$  - analytic in  $D_\epsilon(z_0)$  &  $g(z_0) \neq 0 \Rightarrow$  has Taylor series

$$g(z) = a_0 + a_1(z-z_0) + a_2(z-z_0)^2 + \dots \text{ converging on } D_\epsilon(z_0)$$

$$\Rightarrow \frac{g(z)}{(z-z_0)^m} = \frac{a_0}{(z-z_0)^m} + \frac{a_1}{(z-z_0)^{m-1}} + \dots \text{ converges on } 0 < |z-z_0| < \epsilon$$

$\Rightarrow$  it gives rise to analytic function  $f(z)$  on  $0 < |z-z_0| < \epsilon$  with above Laurent series, which has pole of order  $m$  at  $z_0$ .

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Theorem (Picard's Thm): A function  $f(z)$  with essential singularity at  $z_0$  assumes every complex number, with possibly one exception, as a value in any punctured disk  $D_\varepsilon(z_0) \setminus \{z_0\}$

[ Note: In other words for any  $w \in \mathbb{C}$ , except for possibly one value, and for any  $\varepsilon > 0$  there are actually  $\infty$  many points  $z \in D_\varepsilon(z_0) \setminus \{z_0\}$  with  $f(z) = w$ .

Combining above Lemma 1, Lemma 2, Theorem, we arrive at:

Upshot: One can determine type of singularity of  $f(z)$  at  $z_0$  by looking at the values of  $f(z)$  as  $z \rightarrow z_0$ . Explicitly:

- 1)  $z_0$ -removable singularity  $\Leftrightarrow f(z)$  has a finite limit as  $z \rightarrow z_0$
- 2)  $z_0$ -pole  $\Leftrightarrow f(z) \rightarrow \infty$  as  $z \rightarrow z_0$ .
- 3)  $z_0$ -essential singularity  $\Leftrightarrow f(z)$  has no finite/infinite limit  
 $\Leftrightarrow f(z)$  is neither bounded near  $z_0$  nor  $\rightarrow \infty$  as  $z \rightarrow z_0$

Remark: a) Looking back at Ex 1, we could argue that  $z_0 = 0$  is essential singularity by noting that  $\forall \varepsilon > 0$   $\cos \frac{1}{z}$  varies all the way from  $-1$  to  $1$  as  $z \in (-\varepsilon, \varepsilon) \subseteq \mathbb{R} \subseteq \mathbb{C}$  and has no limit as  $z \rightarrow 0$  there  
b) Note that when viewing  $\cos \frac{1}{z}$  on  $\mathbb{R}$  it is bounded, but once we view it on  $\mathbb{C}$ , then we have Picard's Theorem!

Ex 2: Classify zeroes and singularities of  $f(z) = \frac{\tan z}{z}$

\* Home Extra Reading: Read "Laurent expansion" worksheet (posted in Brightspace)

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► (Solution of Ex2)

$$f(z) = \frac{\tan z}{z} = \frac{\sin z}{\cos z \cdot z}$$

Possible "bad points":  $z=0$  or  $z = \frac{\pi}{2} + \pi k$ ,  $k$ -integer

Case 1:  $z_0=0$

As before  $\frac{\sin z}{z} = 1 - \frac{z^2}{3!} + \frac{z^4}{5!} - \dots$  near  $z_0=0$

while  $\cos z$  is analytic near  $z_0$  and  $\cos(z_0)=1 \neq 0 \Rightarrow \frac{1}{\cos z}$  is analytic near  $z_0$ ,  
i.e. has Taylor series  $a_0 + a_1 z + a_2 z^2 + \dots$

$\Rightarrow f(z) = (1 - \frac{z^2}{3!} + \frac{z^4}{5!} - \dots)(a_0 + a_1 z + a_2 z^2 + \dots)$  has no  $z^{-1}$   $\Rightarrow z_0=0$  - removable singularity

Case 2:  $z_0 = \frac{\pi}{2} + \pi k$

As  $\cos z_0 = 0$  and  $\cos'(z_0) = \pm 1 \neq 0 \Rightarrow$  Taylor series of  $\cos z$  near  $z_0$  reads:

$$\cos z = \underbrace{a_1}_{\neq 0} (z-z_0) + a_2 (z-z_0)^2 + \dots \Rightarrow \frac{1}{\cos z} = \frac{1}{z-z_0} \cdot \frac{1}{\underbrace{a_1 + a_2(z-z_0) + a_3(z-z_0)^2 + \dots}_{\text{analytic near } z_0}}$$

Thus, Laurent series of  $\frac{1}{\cos z}$  near  $z_0$  takes the form:

$$\frac{1}{\cos z} = (z-z_0)^{-1} (b_0 + b_1(z-z_0) + b_2(z-z_0)^2 + \dots) = b_0(z-z_0)^{-1} + b_1 + b_2(z-z_0) + \dots$$

On the other hand,  $\frac{\sin z}{z}$  is analytic near  $z_0$  and  $\frac{\sin z_0}{z_0} \neq 0 \Rightarrow$  its Laurent

series is  $\frac{\sin z}{z} = \underbrace{c_0}_{\neq 0} + c_1(z-z_0) + c_2(z-z_0)^2 + \dots$

So:  $f(z) = (b_0(z-z_0)^{-1} + b_1 + \dots)(\underbrace{c_0}_{\neq 0} + c_1(z-z_0) + \dots) = b_0 c_0 (z-z_0)^{-1} + \text{higher powers}$

$\Rightarrow z_0 = \frac{\pi}{2} + \pi k$  ( $k$ -integer) are poles of order 1, i.e. simple poles

As per zeros: potential zeros are  $z = \pi k$  ( $k$ -integer)

Case 1:  $z_0 = \pi k$ ,  $k \neq 0$ .

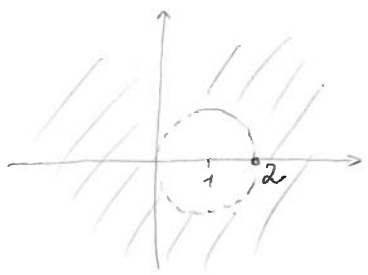
As  $\sin'(z_0) = \pm 1 \neq 0 \Rightarrow \sin z$  has simple zero at  $z_0 \Rightarrow z_0 = \pi k$  ( $k \neq 0$ ) are simple zeros

Case 2:  $z_0 = 0$

Then  $\frac{\sin z}{z} = 1 - \frac{z^2}{3!} + \frac{z^4}{5!} - \dots$  } this is actually an isolated singularity

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Ex 3: Find Laurent expansion of  $f(z) = \frac{z^2 - z + 2}{z - 2}$  in  $D = \{z \in \mathbb{C} : |z - 1| > 1\}$



As  $f(z)$  is analytic in this washer, it does admit Laurent series presentation.

Numerator =  $z^2 - z + 2 = (z - 1)^2 + (z - 1) + 2$

Denominator =  $z - 2 = (z - 1) - 1 \Rightarrow \frac{1}{z - 2} = \frac{1}{(z - 1) - 1}$

Key: Use the geometric progression formula  $\frac{1}{1 - a} = 1 + a + a^2 + \dots$  if  $|a| < 1$

In our case  $|z - 1| > 1$ , so we cannot write above as  $-(1 + (z - 1) + (z - 1)^2 + \dots)$

Instead:  $\frac{1}{(z - 1) - 1} = \frac{1}{(z - 1)(1 - \frac{1}{z - 1})}$  and  $|\frac{1}{z - 1}| < 1 \Rightarrow$  can apply boxed f.l.a:

$$\begin{aligned} \frac{1}{z - 2} &= \frac{1}{(z - 1) - 1} = \frac{1}{z - 1} \left( 1 + \frac{1}{z - 1} + \frac{1}{(z - 1)^2} + \frac{1}{(z - 1)^3} + \dots \right) \\ &= \frac{1}{z - 1} + \frac{1}{(z - 1)^2} + \frac{1}{(z - 1)^3} + \dots \end{aligned}$$

Thus:  $f(z) = ((z - 1)^2 + (z - 1) + 2) \left( \frac{1}{z - 1} + \frac{1}{(z - 1)^2} + \frac{1}{(z - 1)^3} + \dots \right)$

$$\begin{aligned} &= (z - 1) + 1 + \frac{1}{z - 1} + \frac{1}{(z - 1)^2} + \dots \\ &\quad + 1 + \frac{1}{z - 1} + \frac{1}{(z - 1)^2} + \dots \\ &\quad + \frac{2}{z - 1} + \frac{2}{(z - 1)^2} + \dots \end{aligned}$$

$= (z - 1) + 2 + \sum_{n=1}^{\infty} 4(z - 1)^{-n}$