Lecture #27

(Last time before Spring Break, we discussed 3 types of isolated singularities, and how to determine them)

- Let's start from the following problem (50% of old quiz in MA425):

Ex 1: For each of the following function $f(z)$ & point z_0, determine the type of singularity of f at z_0:

a) removable

b) pole

c) essential

d) not an isolated singularity

1) $f(z) = \frac{1-\cos z}{z^2}$, $z_0 = 0$

2) $f(z) = \operatorname{Log} z$, $z_0 = 0$

3) $f(z) = \cot z - \frac{1}{z}$, $z_0 = 0$

4) $f(z) = \frac{z}{e^z+1}$, $z_0 = \pi i$

1) $\cos z = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \dots \Rightarrow 1 - \cos z = \frac{z^2}{2!} - \frac{z^4}{4!} + \dots \Rightarrow f(z) = \frac{1}{2!} - \frac{z^2}{4!} + \dots \quad \forall z \neq 0$

(as radius of convergence $= \infty$)

We note that above series converges wherever Taylor series of $\cos z$ does

So: a) removable singularity

2) $\operatorname{Log} z$ is not even continuous on any $D_\varepsilon(0) \setminus \{0\}$ for $\varepsilon > 0$

So: d) not isolated singularity.

3) $\left.\begin{aligned} \cos z &= 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \dots \\ \sin z &= z - \frac{z^3}{3!} + \dots \end{aligned}\right\} \Rightarrow \cot z = \frac{1 - \frac{z^2}{2!} + \dots}{z - \frac{z^3}{3!} + \dots} = \frac{1 - \frac{z^2}{2!} + \dots}{z\left(1 - \frac{z^2}{3!} + \dots\right)}$

$= \frac{1}{z} \cdot \left(1 - \frac{z^2}{2!} + \dots\right)\left(1 + \frac{z^2}{3!} + \dots\right) = \frac{1}{z} - \frac{1}{3}z + \dots$

$\Rightarrow f(z) = -\frac{1}{3}z + (\dots)$ higher order terms

So: a) removable singularity

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[Warning: Do not apply familiar formula $\sum \frac{z^n}{n!}$ if your $z_0 \neq 0$!]

4) e^z around $z_0 = \pi i$ has Taylor series $\sum_{n\geq 0} \frac{(e^z)^{(n)}}{n!}\Big|_{z=z_0} \cdot (z-z_0)^n$, which is $-1-(z-\pi i)-\frac{1}{2!}(z-\pi i)^2-\ldots \Rightarrow e^z+1 = -(z-\pi i)-\frac{1}{2!}(z-\pi i)^2-\ldots$

Numerator z has Taylor series $\pi i + (z-\pi i)$.

Thus: $f(z) = \dfrac{\pi i + (z-\pi i)}{-(z-\pi i)-\frac{1}{2!}(z-\pi i)^2-\ldots} = -\dfrac{1}{z-\pi i}\left(\pi i + (z-\pi i)\right)\left(1-\frac{1}{2}(z-\pi i)+\ldots\right)$

which clearly has a pole of order 1.

So: b) pole.

Def: Poles of order 1 are usually called simple poles.

A simple observation relates poles/zeroes of f to zeroes/poles of $\frac{1}{f}$:

Lemma 1: a) If f has a zero of order m at z_0, then $\frac{1}{f(z)}$ has a pole of order m at z_0.

b) If f has a pole of order m at z_0, then $\frac{1}{f(z)}$ has a removable singularity at z_0 and if we consider $g(z) = \begin{cases} 1/f(z), & z \neq z_0 \\ 0, & z = z_0 \end{cases}$ then it has a zero of order m at z_0.

a) $f(z) = a_m(z-z_0)^m + a_{m+1}(z-z_0)^{m+1}+\ldots$ – Taylor series expansion

$$\Downarrow \quad \frac{1}{f(z)} = \frac{1}{a_m(z-z_0)^m} \cdot \frac{1}{1+\frac{a_{m+1}}{a_m}(z-z_0)+\frac{a_{m+2}}{a_m}(z-z_0)^2+\ldots}$$

Note: Denominator converges in small nbhd $D_\varepsilon(z_0)$ where $f(z)$ is analytic, and as $z \to z_0$ it tends to 1, hence $\left|\frac{a_{m+1}}{a_m}(z-z_0)+\frac{a_{m+2}}{a_m}(z-z_0)^2+\ldots\right| < 1$ for $|z-z_0|<\varepsilon$ (some $\varepsilon>0$) and we can apply geometric progression to conclude

$$\frac{1}{f(z)} = \frac{1}{a_m(z-z_0)^m}\left(1-\frac{a_{m+1}}{a_m}(z-z_0)+\ldots\right) \Rightarrow \text{has order } m \text{ pole at } z_0$$

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b) If $f(z) = \frac{a_{-m}}{(z-z_0)^m} + \frac{a_{1-m}}{(z-z_0)^{m-1}} + \dots$ with $a_{-m} \neq 0$, then

writing $f(z) = \frac{a_{-m}}{(z-z_0)^m}\left(1 + \frac{a_{1-m}}{a_{-m}}(z-z_0) + \dots\right)$ we again use geometric progression series to conclude

$$\frac{1}{f(z)} = \frac{(z-z_0)^m}{a_{-m}}\left(1 - \frac{a_{1-m}}{a_{-m}}(z-z_0) + \dots\right) \quad \text{for } z \in D_\varepsilon(z_0)\setminus z_0$$

ε-small enough so that f-analytic there

As $f(z_0)$ is not well-defined, so is $\frac{1}{f}(z_0)$, but the above implies that $g(z) := \begin{cases} 1/f(z), & \forall z \in D_\varepsilon(z_0)\setminus z_0 \\ 0, & z = z_0 \end{cases}$ is analytic on $D_\varepsilon(z_0)$ and has order m zero at z_0 ■

Ex2: Classify zeroes & singularities of $\frac{\sin z}{(z^2-\pi^2)^3}$.

- Zeroes: possible at $z = \pi k$, $k \in \mathbb{Z}$
- Poles: possible at $z = \pm\pi$.

$\Rightarrow$ $\boxed{\text{At } z = \pi k \text{ with } k \in \mathbb{Z}\setminus\{\pm 1\} \text{ have zeroes and } \cos(\pi k) \neq 0 \Rightarrow \text{order } 1}$

Warning: As both $z_0 = \pi$ and $z_0 = -\pi$ appear in both lists, we need to take a closer look there!

Case $z_0 = \pi$

Taylor series of $\sin z$ around z_0 is $\sin z_0 + \frac{\sin' z_0}{1!}(z-z_0) + \dots$

For $z_0 = \pi$: $\cos \pi = -1$ $\Rightarrow \sin z = -(z-\pi) + \dots$ higher powers of $z-\pi$

Taylor series of $(z^2-\pi^2)^3$ around $z_0 = \pi$ is $(z-\pi)^3 \times \left(\begin{array}{c}\text{Taylor series}\\ \text{of } (z+\pi)^3 \text{ at } z_0\end{array}\right)$

$\Rightarrow$ denominator $= 8\pi^3(z-\pi)^3 + (\dots)$ higher powers of $z-\pi$ $8\pi^3 + *(z-\pi) + (\dots)$

So: $f(z) = \frac{\sin z}{(z^2-\pi^2)^3}$ which is analytic in $D_\varepsilon(\pi)\setminus\{\pi\}$ has Laurent series

$$\frac{-(z-\pi) + (\dots)}{8\pi^3(z-\pi)^3(1+\dots)} = -\frac{1}{8\pi^3}(z-\pi)^{-2} + (\dots) \text{ higher powers of } (z-\pi)$$

$\Rightarrow$ $\boxed{z_0 = \pi - \text{pole of order 2!}}$

Case $z_0 = -\pi$: Similar analysis $\Rightarrow$ $\boxed{-\pi - \text{pole of order 2!}}$

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The above example illustrates the following rule:

Claim: Assume $f(z) = \frac{g(z)}{h(z)}$ with $g(z), h(z)$ both analytic at disk $D_R(z_0)$ for some $R>0$, and assume $g(z_0) = 0 = h(z_0)$. Let m be the order of zero of $g(z)$ at z_0, and n be the order of zero of $h(z)$ at z_0. Then:

- if $n > m$: $f(z)$ has a pole of order $n-m$ at z_0
- if $m \geq n$: $f(z)$ has a removable singularity at z_0 and regularizing at z_0 via $\tilde{f}(z) = \begin{cases} f(z) & \text{if } z \neq z_0 \\ \lim_{u\to z_0} f(u) & \text{if } z = z_0 \end{cases}$ has a zero of order $m-n$ at z_0

Also note that when inverting Taylor series, like we did in Ex 1(3), there are two ways to proceed:

- use geometric progression formula
- reconstruct term by term

e.g. $$\cot z = \frac{1 - \frac{z^2}{2} + \frac{z^4}{24} - \dots}{z\left(1 - \frac{z^2}{6} + \frac{z^4}{120} - \dots\right)} = \frac{1}{z}\left(1 - \frac{z^2}{2} + \frac{z^4}{24} - \dots\right) \cdot \frac{1}{1 - \frac{z^2}{6} + \frac{z^4}{120} - \dots}$$

Let's write $$\frac{1}{1 - \frac{z^2}{6} + \frac{z^4}{120} - \dots} = a_0 + a_1 z + a_2 z^2 + a_3 z^3 + a_4 z^4 + \dots$$

$$\Downarrow$$

$$1 = \underbrace{\left(1 - \frac{z^2}{6} + \frac{z^4}{120} - \dots\right)\left(a_0 + a_1 z + a_2 z^2 + a_3 z^3 + a_4 z^4 + \dots\right)}_{\text{open brackets}}$$

$$= a_0 + a_1 \cdot z + \left(a_2 - \frac{a_0}{6}\right) z^2 + \left(a_3 - \frac{a_1}{6}\right) z^3 + \left(a_4 - \frac{a_2}{6} + \frac{a_0}{120}\right) z^4 + \dots$$

Equating coefficients of each z-power, we get:

$a_0 = 1$, $a_1 = 0$, $a_2 - \frac{a_0}{6} = 0 \Rightarrow a_2 = \frac{1}{6}$, $a_3 - \frac{a_1}{6} = 0 \Rightarrow a_3 = 0$, $a_4 - \frac{a_2}{6} + \frac{a_0}{120} = 0 \Rightarrow a_4 = \frac{1}{36} - \frac{1}{120}$

$$\Rightarrow \cot z = \frac{1}{z}\left(1 - \frac{z^2}{2} + \frac{z^4}{24} - \dots\right)\left(1 + \frac{z^2}{6} + \frac{7}{360} z^4 + \dots\right) = \frac{1}{z} - \frac{1}{3} z + \left(\frac{7}{360} - \frac{1}{12} + \frac{1}{24}\right) z^3 + \dots$$

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Ex 3: For $f(z)=\frac{1}{(z-1)(z-2)}$ find its Laurent series in:

a) $|z|<1$

b) $1<|z|<2$

c) $|z|>2$

Step 1 (partial fraction decomposition): $\frac{1}{(z-1)(z-2)} = \frac{1}{1-z} - \frac{1}{2-z}$.

a) if $|z|<1$, then by geometric progression:

$$\frac{1}{1-z} = 1+z+z^2+\dots$$

but also $\left|\frac{z}{2}\right|<1 \Rightarrow \frac{1}{2-z} = \frac{1}{2}\cdot\frac{1}{1-\frac{z}{2}} = \frac{1}{2}\left(1+\frac{z}{2}+\frac{z^2}{4}+\dots\right)$

So: $\boxed{f(z) = (1+z+z^2+\dots) - \left(\frac{1}{2}+\frac{1}{4}z+\frac{1}{8}z^2+\dots\right) = \sum_{n=0}^{\infty}\left(1-\frac{1}{2^{n+1}}\right)z^n}$

b) if $1<|z|<2$, then we still have $\frac{1}{2-z} = \frac{1}{2}+\frac{1}{4}z+\frac{1}{8}z^2+\dots$

but $\frac{1}{1-z} \neq 1+z+z^2+\dots$ as the latter sum diverges!

Instead, we write $\frac{1}{1-z} = \frac{1}{-z\left(1-\frac{1}{z}\right)} = -\frac{1}{z}\left(1+\frac{1}{z}+\frac{1}{z^2}+\dots\right) = -\frac{1}{z}-\frac{1}{z^2}-\frac{1}{z^3}-\dots$

So: $\boxed{f(z) = \dots - \frac{1}{z^3} - \frac{1}{z^2} - \frac{1}{z} - \frac{1}{2} - \frac{1}{4}z - \frac{1}{8}z^2 - \dots}$

c) if $|z|>2$, then we use $\frac{1}{1-z} = -\frac{1}{z}-\frac{1}{z^2}-\frac{1}{z^3}-\dots$ as in b), but now have to change $\frac{1}{2-z}$ (as $\left|\frac{z}{2}\right|>1$ and so geom. progr. diverges)

To do so, we write

$$\frac{1}{2-z} = \frac{1}{-z\left(1-\frac{2}{z}\right)} \underset{\left|\frac{2}{z}\right|<1 \text{ now!}}{=} -\frac{1}{z}\left(1+\frac{2}{z}+\frac{4}{z^2}+\frac{8}{z^3}+\dots\right) = -\frac{1}{z}-\frac{2}{z^2}-\frac{4}{z^3}-\dots$$

So: $\boxed{f(z) = \left(-\frac{1}{z}-\frac{1}{z^2}-\frac{1}{z^3}-\dots\right) - \left(-\frac{1}{z}-\frac{2}{z^2}-\frac{4}{z^3}-\dots\right) = \sum_{n\geq 2}\frac{2^{n-1}-1}{z^n}}$

[**Moral:** Two ways to write $\frac{1}{\lambda-\mu} = \frac{1}{\lambda}\cdot\frac{1}{1-\mu/\lambda}$ or $\frac{1}{\lambda-\mu} = -\frac{1}{\mu}\cdot\frac{1}{1-\lambda/\mu}$
and we choose 1st if $|\mu/\lambda|<1$ and second if $|\lambda/\mu|<1$]

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We conclude with the following "tricky" exercise on R.O.C:

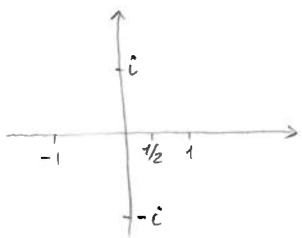
Ex4: Find R.O.C. for Taylor series expansions:

a) $\sum_{n=0}^{\infty} a_n (z-\frac{1}{2})^n = \frac{1}{z^4-1}$

b) $\sum_{n=0}^{\infty} a_n (z-\frac{1}{2})^n = \frac{z-1}{z^4-1}$

c) $\sum_{n=0}^{\infty} a_n z^n = \frac{\mathrm{Log}(z+4)}{z+3}$

a)



By Lectures 22-23, we know that R.O.C. of the series $\sum a_n (z-z_0)^n$ equals the biggest $R>0$ s.t. the result is analytic on $D_R(z_0)$.

As function $\frac{1}{z^4-1}$ has poles at $\pm 1, \pm i$, we see that R = smallest of distances from $z_0 = 1/2$ to $\pm 1, \pm i$

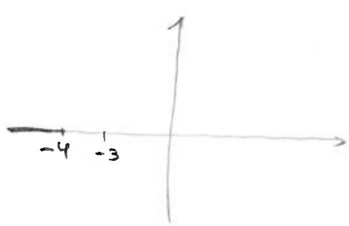
$\Rightarrow$ $R = 1/2$

b) Same picture BUT note that $\frac{z-1}{z^4-1}$ has a removable singularity at 1 and poles at $-1, \pm i$ as we can explicitly write $\frac{z-1}{z^4-1} = \frac{1}{z^3+z^2+z+1}$

Thus: R = smallest of distances from $z_0 = 1/2$ to points $\{i, -i, -1\}$

$\Rightarrow$ $R = \sqrt{5}/2$ [Warning: common mistake is to say $R = 1/2$]

c)



Numerator $= \mathrm{Log}(z+4)$ is analytic in $D_4(0)$ but is discontinuous in $D_R(0)$ for all $R>4$

Also $\frac{1}{z+3}$ has a simple pole at $z=-3$.

However, $\mathrm{Log}(z+4)|_{z=-3} = \mathrm{Log}\,1 = 0 \Rightarrow z=-3$ is a removable singularity of $f(z) = \frac{\mathrm{Log}(z+4)}{z+3}$

$\Rightarrow$ $R = 4$ [Warning: common mistake is to say $R=3$]