Lecture #28

Today: Start Chapter 6 "Residue Theory"

Def: A residue of $f(z)$ at its isolated singularity z_0 is defined as:

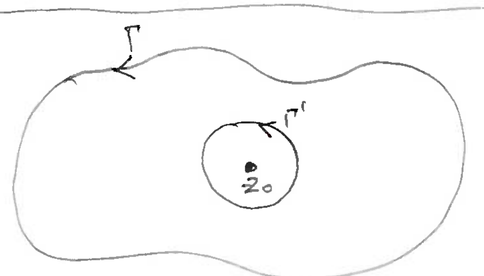
$$\boxed{Res_{z_0}(f) := a_{-1} = \text{coefficient of } (z-z_0)^{-1} \text{ in Laurent series}}$$

- In your book, the same is also denoted by $Res(f; z_0)$ or $Res(z_0)$
- Note that it's common to divide each Laurent series in two parts:

$$\underbrace{\sum_{n=0}^{\infty} a_n (z-z_0)^n}_{\text{"regular part"}} + \underbrace{\sum_{n=1}^{\infty} \frac{a_{-n}}{(z-z_0)^n}}_{\text{"principal part"}}$$

Theorem: If Γ is a positively oriented simple closed curve, point z_0 is inside Γ, and $f(z)$ - analytic on and inside Γ besides point z_0, then

$$\oint_\Gamma f(z)dz = 2\pi i \cdot Res_{z_0}(f)$$



Let $\Gamma' = C_\varepsilon(z_0)$ be a positively oriented circle of radius ε centered at z_0

By Cauchy Theorem: $\oint_\Gamma f(z)dz = \oint_{\Gamma'} f(z)dz$

But $f(z) = \sum_{j=0}^{\infty} a_j (z-z_0)^j + \sum_{j=1}^{\infty} a_{-j}(z-z_0)^{-j}$ with both sums converging uniformly on Γ'

$\Downarrow$

$$\oint_{\Gamma'} f(z)dz = \sum_{j=-\infty}^{\infty} a_j \oint_{\Gamma'} (z-z_0)^j dz \overset{\text{first calculated in Lecture 13}}{=} a_{-1} \cdot 2\pi i = 2\pi i \cdot Res_{z_0}(f)$$

Ex1 (= Ex 7 of p. 314): $\oint_{|z|=1} e^{1/z} \sin(1/z) dz = ?$

$e^{\frac{1}{z}} = 1 + \frac{1}{z} + \frac{1}{2!}\cdot\frac{1}{z^2} + \dots$, $\sin\left(\frac{1}{z}\right) = \frac{1}{z} - \frac{1}{3!}\cdot\frac{1}{z^3} + \dots \Rightarrow Res_0\left(e^{\frac{1}{z}}\sin\left(\frac{1}{z}\right)\right) = 1$

$\Rightarrow$ Answer: $2\pi i$

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Ex2: a) Find residues of $f(z) = z^3 \cos^2\left(\frac{2}{z}\right)$ at all singular points.

b) Evaluate $\oint\limits_{\substack{|z|=3 \\ \text{(posit. orient.)}}} z^3 \cos^2\left(\frac{2}{z}\right) dz$.

▻ a) The only singular point is $z=0$.

Recall $\cos w = 1 - \frac{w^2}{2!} + \frac{w^4}{4!} - \ldots = 1 - \frac{w^2}{2} + \frac{w^4}{24} - \ldots$

$\Rightarrow \cos\left(\frac{2}{z}\right) = 1 - \frac{2}{z^2} + \frac{2}{3} \cdot \frac{1}{z^4} - \ldots \Rightarrow \cos^2\left(\frac{2}{z}\right) = \left(1 - \frac{2}{z^2} + \frac{2}{3} \cdot \frac{1}{z^4} + \ldots\right)\left(1 - \frac{2}{z^2} + \frac{2}{3} \cdot \frac{1}{z^4} + \ldots\right)$

$\underset{\text{open brackets}}{=} 1 - \frac{4}{z^2} + \frac{2/3 + 2/3 + 4}{z^4} + \ldots$

$\Rightarrow f(z) = z^3 \cos^2\left(\frac{2}{z}\right) = z^3 - 4z + \left(\frac{2}{3} + \frac{2}{3} + 4\right) z^{-1} + \ldots$

$\Rightarrow \mathrm{Res}_0 f = \frac{4}{3} + 4 = \frac{16}{3}$

b) As the singular point is inside the positively oriented closed contour, get:

$$\oint\limits_{|z|=3} z^3 \cos^2\left(\frac{2}{z}\right) dz = 2\pi i \cdot \mathrm{Res}_0 f = \frac{32}{3}\pi i$$

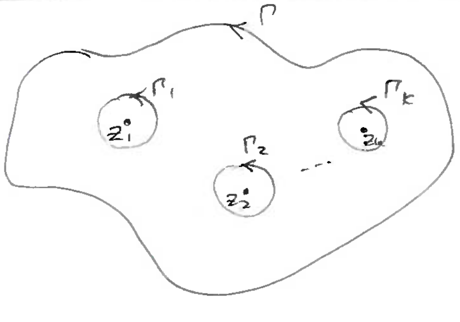
■

Now, the result of previous Theorem naturally generalizes to the case when $f(z)$ is analytic on & inside Γ except for several points (not one):

Theorem ("Cauchy's Residue Theorem"): If Γ is a positively oriented simple closed contour and $f(z)$ is analytic on & inside Γ except for possibly points $z_1, z_2, \ldots, z_k$, then

$$\boxed{\oint\limits_{\Gamma} f(z)dz = 2\pi i \cdot \sum_{j=1}^{k} \mathrm{Res}_{z_j}(f)}$$

▻



Cauchy's thm $\Rightarrow \oint\limits_{\Gamma} f(z)dz = \left(\oint\limits_{\Gamma_1} + \ldots + \oint\limits_{\Gamma_k}\right) f(z)dz$

but $\oint\limits_{\Gamma_j} f(z)dz = 2\pi i \, \mathrm{Res}_{z_j}(f)$

Γ_j – positively oriented circle around z_j.

■

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Ex 3: $\oint_{|z|=1} \frac{1}{z^2 \sin z} dz = ?$

▶ The function $f(z) = \frac{1}{z^2 \sin z}$ is analytic everywhere except for $z = \pi k$ ($k \in \mathbb{Z}$) integers.
However, only for $k=0$ the corresponding value $z_1 = 0$ is inside unit circle!

$$\sin z = z - z^3/3! + z^5/5! - \dots \Rightarrow \frac{1}{\sin z} = \frac{1}{z(1 - \frac{z^2}{6} + \dots)} = \frac{1}{z}\left(1 + \frac{z^2}{6} + \dots\right)$$

$$\Rightarrow \frac{1}{z^2 \sin z} = \frac{1}{z^2} \cdot \frac{1}{z} \cdot \left(1 + \frac{z^2}{6} + \dots\right) \Rightarrow \mathrm{Res}_{z_1}(f) = \frac{1}{6}$$

So: $\oint_{|z|=1} \frac{1}{z^2 \sin z} dz = 2\pi i \cdot \frac{1}{6} = \frac{\pi i}{3}$

■

Note: If we were asked to evaluate $\oint_{|z|=4} \frac{1}{z^2 \sin z} dz$, then would also need to add $2\pi i \left(\mathrm{Res}_{\pi} f + \mathrm{Res}_{-\pi} f\right)$

Note that if $f(z)$ has a removable singularity at z_0, then $\mathrm{Res}_{z_0} f = 0$.
On the other hand if $f(z)$ has a simple pole at z_0, i.e.

$$f(z) = \frac{a_{-1}}{z - z_0} + a_0 + a_1(z - z_0) + a_2(z - z_0)^2 + \dots$$

$$\Downarrow$$

$$(z - z_0) f(z) = a_{-1} + a_0(z - z_0) + a_1(z - z_0)^2 + \dots$$

$$\Downarrow$$

(*) $\boxed{\mathrm{Res}_{z_0}(f) = a_{-1} = \lim_{z \to z_0} \left((z - z_0) f(z)\right)}$ ← if z_0 - simple pole of $f(z)$

Lemma 1: If $f(z) = \frac{p(z)}{q(z)}$ with p, q - analytic at z_0, $p(z_0) \neq 0$, z_0 - simple zero of $q(z)$, then $\boxed{\mathrm{Res}_{z_0}(f) = \frac{p(z_0)}{q'(z_0)}}$

▶ By above:

$$\mathrm{Res}_{z_0}(f) = \lim_{z \to z_0} \frac{p(z) \cdot (z - z_0)}{q(z)} = \lim_{z \to z_0} \frac{p(z)}{\frac{q(z) - q(z_0)}{z - z_0}} = \frac{p(z_0)}{q'(z_0)}$$

■

This simple lemma is very helpful as we shall now see.

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Ex 4: Evaluate $\oint_{|z|=5} \frac{\sin z}{z^2-4}\,dz$.

▶ Function $f(z)=\frac{\sin z}{z^2-4}$ has two simple poles at $z_1=2$, $z_2=-2$, both of which lie inside our contour.

$\mathrm{Res}_{z_1} f = \frac{\sin 2}{2\cdot 2} = \frac{\sin 2}{4}$, $\mathrm{Res}_{z_2} f = \frac{\sin(-2)}{2(-2)} = \frac{\sin 2}{4}$ by above Lemma

(alternatively could also apply formula (*))

So: $\oint_{|z|=5} f(z)\,dz = 2\pi i\left(\frac{\sin 2}{4}+\frac{\sin 2}{4}\right) = \sin 2\cdot \pi i$

$\square$

Ex 5: a) Find the residues of $f(z)=\frac{1}{z^2+z+1}$ at all singular points

b) Compute $\oint_{|z|=8} \frac{1}{z^2+z+1}\,dz$

▶ a) The only singular points of $f(z)$ are zeroes of z^2+z+1

$z^2+z+1=(z+\frac{1}{2})^2+\frac{3}{4} \Rightarrow$ roots are $-\frac{1}{2}\pm\frac{\sqrt{3}}{2}i$ (or just use $z=\frac{-1\pm\sqrt{-3}}{2}$)

By Lemma 1 above:

$$\mathrm{Res}_{-\frac{1}{2}+\frac{\sqrt{3}}{2}i} f = \left.\frac{1}{2z+1}\right|_{z=-\frac{1}{2}+\frac{\sqrt{3}}{2}i} = \frac{1}{\sqrt{3}i} = -\frac{i}{\sqrt{3}}$$

$$\mathrm{Res}_{-\frac{1}{2}-\frac{\sqrt{3}}{2}i} f = \left.\frac{1}{2z+1}\right|_{z=-\frac{1}{2}-\frac{\sqrt{3}}{2}i} = \frac{1}{-\sqrt{3}i} = \frac{i}{\sqrt{3}}$$

b) $\oint_{|z|=8} \frac{1}{z^2+z+1}\,dz = 2\pi i\left(\frac{1}{\sqrt{3}i}-\frac{1}{\sqrt{3}i}\right)=0$

$\square$

Q: What if z_0 is a pole of order $m>1$? Then:

$$(z-z_0)^m f(z) = a_{-m} + a_{1-m}(z-z_0) + \_\_\_ + a_{-1}(z-z_0)^{m-1} + a_0(z-z_0)^m + \dots$$

$$\Downarrow$$

$$\boxed{\mathrm{Res}_{z_0} f = a_{-1} = \frac{1}{(m-1)!}\left[\left(\frac{d}{dz}\right)^{m-1}\Big(\underbrace{(z-z_0)^m f(z)}_{\text{more precise: analytic continuation to } z_0}\Big)\right]_{|z=z_0} = \frac{1}{(m-1)!}\lim_{z\to z_0}\left((z-z_0)^m f(z)\right)^{(m-1)}}$$

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Ex 6: Find the residues of $f(z)=\frac{e^{2z}}{z^2(z^2-1)}$ at all singular points.

▸ Clearly, singular points are: $\underbrace{z_1=1,\ z_2=-1,}_{\text{simple poles}}$ $z_3=0$.

$$\mathrm{Res}_{z_1}(f)=\frac{e^{2z}}{z^2(z+1)}\Big|_{z=1}=\underline{\underline{\frac{e^2}{2}}}$$

$$\mathrm{Res}_{z_2}(f)=\frac{e^{2z}}{z^2(z-1)}\Big|_{z=-1}=\underline{\underline{\frac{e^{-2}}{-2}}}$$

(For this calculation of residues, we just factorized $z^2-1=(z-1)(z+1)$, but we could also apply Lemma 1)

$$\mathrm{Res}_{z_3}(f)=\lim_{z\to 0}\frac{d}{dz}\left(\frac{e^{2z}}{z^2-1}\right)=\left[\frac{e^{2z}\cdot 2\cdot(z^2-1)-e^{2z}\cdot 2z}{(z^2-1)^2}\right]\Big|_{z=0}=\underline{\underline{-2}}$$

§6.2 Trigonometric integrals

The following problem does not seem to involve any cpx analysis:

Problem: Evaluate $\int_0^{2\pi}\frac{1}{2+\sin\theta}\,d\theta$ ← looks like a usual calculus problem (but not too simple!)

Before we illustrate on this example, let's present the key idea.

Let Γ be the unit circle positively oriented, which can be parametrized by $z=e^{i\theta}=\cos\theta+i\sin\theta$ with $0\le\theta\le 2\pi$.

Then: $\cos\theta=\frac{z+z^{-1}}{2}$, $\sin\theta=\frac{z-z^{-1}}{2i}$, $dz=e^{i\theta}\cdot i\,d\theta=iz\,d\theta\Rightarrow d\theta=\frac{dz}{iz}$

Hence, for any function $U(\cdot,\cdot)$, we can reduce:

$$\boxed{\int_0^{2\pi}U(\cos\theta,\sin\theta)\,d\theta=\oint_\Gamma U\left(\frac{z+1/z}{2},\ \frac{z-1/z}{2i}\right)\frac{dz}{iz}}$$

In practice (at least for §6.2), we shall always have $U(\cdot,\cdot)$ being rational f-n of 2 arguments, so that we integrate rat-l f-n of z over Γ, which we now know how to do!

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Solution of Problem

$$\int_0^{2\pi} \frac{d\theta}{2+\sin\theta} = \oint_{|z|=1} \frac{1}{2+\frac{z-1/z}{2i}} \cdot \frac{1}{iz}\,dz = \oint_{|z|=1} \frac{2}{z^2+4iz-1}\,dz$$

Function $f(z) = \dfrac{2}{z^2+4iz-1}$ has singularities only at roots of $z^2+4iz-1$ which are $\dfrac{-4i \pm \sqrt{-16+4}}{2} = (-2\pm\sqrt{3})i$.

However, $(-2-\sqrt{3})i$ is outside of the unit circle
$(-2+\sqrt{3})i$ is inside the unit circle $\Big\} \to$

$$\Rightarrow \oint_{|z|=1} \frac{2}{z^2+4iz-1}\,dz = 2\pi i \cdot \underbrace{\operatorname{Res}_{(-2+\sqrt{3})i} \frac{2}{z^2+4iz-1}}_{\overset{\text{Lemma 1}}{=}\ \frac{2}{2(-2+\sqrt{3})i+4i} = \frac{1}{\sqrt{3}\,i}} = \boxed{\frac{2\pi}{\sqrt{3}}}$$

Note: If you got not a real number that indicates you made an error as the original integral must clearly be real!