Lecture #29

* Last time:

§6.1 → Cauchy's Residue Theorem: $\boxed{\displaystyle\oint_{\Gamma} f(z)dz = 2\pi i \cdot \sum_{\substack{z_j - \text{singular pts} \\ \text{of } f(z) \text{ inside } \Gamma}} \operatorname{Res}_{z_j}(f)}$

Γ positively oriented simple contour

[Note: This result is very straightforward and just requires to carefully compute residues!

§6.2 → Trigonometric Integrals

Key: $$\boxed{\int_0^{2\pi} U(\cos\theta, \sin\theta)d\theta = \underbrace{\oint_{|z|=1} U\left(\frac{z+1/z}{2}, \frac{z-1/z}{2i}\right)\cdot\frac{1}{iz}dz}_{\text{evaluated now via §6.1}}}$$

[Note: Key was to reduce real functions integral to the one of complex f-n integral over closed contour

Ex1: Evaluate $I = \int_0^{\pi} \frac{8}{5+2\cos\theta}d\theta$

► While the above integral is $\int_0^{\pi}$ and not $\int_0^{2\pi}$, it can be easily reduced to the latter. Namely, recall that $\cos(-\theta)=\cos(\theta) \Rightarrow \int_{-\pi}^{0} \frac{8}{5+2\cos\theta}d\theta = I$

So: $\boxed{2I = \int_{-\pi}^{\pi} \frac{8}{5+2\cos\theta}d\theta}$. Now we can apply above reasoning, with unit circle $|z|=1$ parametrized via $z=e^{i\theta}$, $-\pi\le\theta\le\pi$:

$$\int_{-\pi}^{\pi}\frac{8}{5+2\cos\theta}d\theta = \oint_{|z|=1}\frac{8}{5+2\cdot\frac{z+z^{-1}}{2}}\cdot\frac{dz}{iz} = \frac{8}{i}\oint_{|z|=1}\frac{1}{z^2+5z+1}dz$$

The function $f(z) = \frac{1}{z^2+5z+1}$ has simple poles at $z = \frac{-5\pm\sqrt{21}}{2}$ (and no other singular pts)

However, $\left|\frac{-5-\sqrt{21}}{2}\right| > 1 > \left|\frac{-5+\sqrt{21}}{2}\right| \Rightarrow$ need only to evaluate $\operatorname{Res}_{z_1=\frac{-5+\sqrt{21}}{2}} f(z)$

But as pole is simple: $\operatorname{Res}_{z_1} f(z) = \left.\frac{1}{2z+5}\right|_{z=z_1} = \frac{1}{\sqrt{21}}$

Thus: $2I = \int_{-\pi}^{\pi}\frac{8}{5+2\cos\theta}d\theta = \frac{8}{i}\cdot 2\pi i\cdot\frac{1}{\sqrt{21}} = \frac{16\pi}{\sqrt{21}} \Rightarrow I = \frac{8\pi}{\sqrt{21}}$

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For the rest of this week, we shall be evaluating real-type integrals and the key part will be always to choose appropriate "closure" of line segment / ray / all $\mathbb{R}$ to a simple closed curve.

* **Today**: $\boxed{\text{§6.3 Improper Integrals}}$

Recall: $\int_0^{+\infty} f(x)dx = \lim_{R\to+\infty} \int_0^R f(x)dx$ if exists

$\int_{-\infty}^0 f(x)dx = \lim_{R\to+\infty} \int_{-R}^0 f(x)dx$ if exists

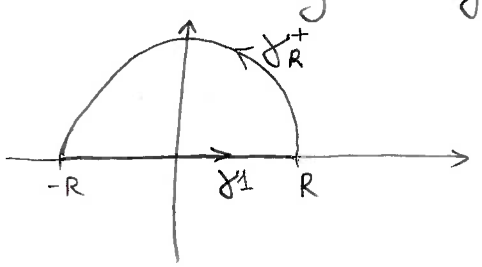
$\rightsquigarrow \int_{-\infty}^{+\infty} f(x)dx = \left(\int_0^{+\infty} + \int_{-\infty}^0\right) f(x)dx$ if both exist.

Note: If $\int_{-\infty}^{+\infty} f(x)dx$ exists then it can be also evaluated as $\lim_{R\to+\infty}\int_{-R}^{R} f(x)dx$. However, the latter may exist while $\int_{-\infty}^{+\infty} f(x)dx$ is not defined, e.g. $f(x)=x$!

Def: $\boxed{\text{p.v.}\int_{-\infty}^{+\infty} f(x)dx := \lim_{R\to+\infty}\int_{-R}^{R} f(x)dx}$ is called Cauchy's principal value.

Ex2: Evaluate $I = \int_{-\infty}^{+\infty} \frac{dx}{x^4+1} = \text{p.v.}\int_{-\infty}^{+\infty}\frac{dx}{x^4+1}$

Key: Close line segment γ_1 from $-R$ to $+R$ by half-circle γ_R^+ in $\{\operatorname{Im} z \geq 0\}$



- Compute $\int_{\gamma_1+\gamma_R^+}\frac{dz}{z^4+1}$ via Cauchy's Residue Thm
- Show $\lim_{R\to+\infty}\int_{\gamma_R^+}\frac{dz}{z^4+1} = 0$

► $\frac{1}{z^4+1}$ has singular points (all being simple poles) at $e^{\pm i\cdot\frac{\pi}{4}}, e^{\pm i\cdot\frac{3\pi}{4}}$. But inside above curve $\gamma_1+\gamma_R^+$ only $e^{i\cdot\frac{\pi}{4}}, e^{i\cdot\frac{3\pi}{4}}$ are contained. Also:

$$\operatorname{Res}_{e^{i\frac{\pi}{4}}}\frac{1}{z^4+1} = \frac{1}{4z^3}\Big|_{z=e^{i\pi/4}} = \frac{1}{4}e^{i\cdot\frac{-3\pi}{4}} = \frac{1}{4}\cdot\frac{-1-i}{\sqrt{2}}$$

$$\operatorname{Res}_{e^{i\cdot\frac{3\pi}{4}}}\frac{1}{z^4+1} = \frac{1}{4z^3}\Big|_{z=e^{i\cdot3\pi/4}} = \frac{1}{4}e^{i\cdot\frac{-9\pi}{4}} = \frac{1}{4}\cdot\frac{1-i}{\sqrt{2}}$$

So: $\int_{\gamma_1}\frac{dz}{z^4+1} + \int_{\gamma_R^+}\frac{dz}{z^4+1} = 2\pi i\cdot\frac{1}{4}\cdot\left(\frac{-1-i}{\sqrt{2}}+\frac{1-i}{\sqrt{2}}\right) = \pi/\sqrt{2}$

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(Continuation)

► Now, taking $R \to +\infty$, we get $\lim_{R\to+\infty} \int_{\gamma_1} \frac{dz}{z^4+1} = I$.

<u>Claim</u>: $\lim_{R\to+\infty} \int_{\gamma_R^+} \frac{dz}{z^4+1} = 0$

► $\left| \int_{\gamma_R^+} \frac{dz}{z^4+1} \right| \leq \text{length}(\gamma_R^+) \cdot \max_{z\in\gamma_R^+} \left| \frac{1}{z^4+1} \right| \leq \pi R \cdot \frac{1}{R^4-1} \xrightarrow{R\to+\infty} 0$ ∎

<u>Thus</u>: $I = \pi/\sqrt{2}$ ∎

<u>Remark</u>: The above choice of γ_R^+ is going to work as long as:

a) $\int_{\gamma_R} f(z)dz \longrightarrow 0$ as $R \to +\infty$

b) $f(z)$ has finitely many singular points in the upper half plane $\mathcal{H} = \{z : \operatorname{Im} z > 0\}$

The above condition a), in particular, holds for certain rational f-s:

<u>Lemma 1</u>: If $f(z) = \frac{P(z)}{Q(z)}$ - ratio of two polynomials with $\deg Q \geq \deg P + 2$

then $\int_{\gamma_R^+} f(z)dz \longrightarrow 0$ as $R \to +\infty$

► If $P(z) = a_0 + a_1 z + \_ + a_m z^m$, $Q(z) = b_0 + b_1 z + \_\_ + b_n z^n$ with $n \geq m+2$, $a_m \neq 0$, $b_n \neq 0$

then $\left| \frac{P(z)}{Q(z)} \right| \leq \left(\left| \frac{a_m}{b_n} \right| + \varepsilon \right) \cdot \frac{1}{|z|^{n-m}}$ $\forall \varepsilon > 0$ as long as $|z| \gg 0$. Then:

$\left| \int_{\gamma_R^+} \frac{P(z)}{Q(z)} dz \right| \leq \pi R \cdot \left(\left| \frac{a_m}{b_n} \right| + 1 \right) \cdot \frac{1}{R^{n-m}} \xrightarrow{R\to+\infty} 0$ ∎

<u>Note</u>: Same argument would also apply if we used $\gamma_R^- =$ half circle from R to $-R$ in lower half plane $-\mathcal{H}$ (will be useful in §6.4!)

<u>BUT</u>: $\gamma_1 + \gamma_R^-$ is negatively oriented ⇒ don't forget extra minus sign!

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Ex 3: Evaluate $I = \int_0^{+\infty} \frac{x^2}{(x^2+1)(x^2+4)} dx$

Step 0: As function $f(x) = \frac{x^2}{(x^2+1)(x^2+4)}$ is even, i.e. $f(-x) = f(x)$, we get (similar to Ex 1) that $2I = \int_{-\infty}^{+\infty} \frac{x^2}{(x^2+1)(x^2+4)} dx$

(↑ can you explain why it's well-defined?)

We shall again close γ_1 from $-R$ to $+R$ with a half circle γ_R^+ in $\mathcal{H}$.

Step 1: Cauchy's Residue Thm:

$$\underbrace{\int_{\gamma_1} f(z)dz}_{\substack{\downarrow R\to+\infty \\ 2I}} + \underbrace{\int_{\gamma_R^+} f(z)dz}_{\substack{\downarrow R\to+\infty \\ 0 \text{ by Lemma 1}}} = 2\pi i \cdot \sum_{\substack{z_j\text{-singular} \\ \text{of } f(z) \text{ inside } \gamma_1+\gamma_R}} \operatorname{Res}_{z_j}(f)$$

Step 2: Residue computation

$f(z)$ has 4 singular points (all simple poles): $\pm i$, $\pm 2i$. But only i & $2i$ will lie inside for $R > 2$.

$$\left.\begin{aligned} \operatorname{Res}_i f(z) &= \left.\frac{z^2}{(z^2+4)(z+i)}\right|_{z=i} = -\frac{1}{6i} \\ \operatorname{Res}_{2i} f(z) &= \left.\frac{z^2}{(z^2+1)(z+2i)}\right|_{z=2i} = \frac{1}{3i} \end{aligned}\right\} \Rightarrow \sum_{z_j\text{-singular inside}} \operatorname{Res}_{z_j}(f) = \frac{1}{3i} - \frac{1}{6i} = \frac{1}{6i}$$

Step 3: $2I = 2\pi i \cdot \frac{1}{6i} \Rightarrow I = \pi/6$ ⊠

! **However**, it's not always the half circle that we shall take.

The following example is presented in the end of §6.3 of your book.

Ex 4: Evaluate $I = \text{p.v.} \int_{-\infty}^{+\infty} \frac{e^{ax}}{1+e^x} dx$ with $0 < a < 1$

Issues with γ_R: 1) ∞ many singular points in $\pm\mathcal{H}$ (as poles are $(2k+1)\pi i$)

2) no easy way to estimate $\int_{\gamma_R^+} \xrightarrow[R\to+\infty]{} 0$

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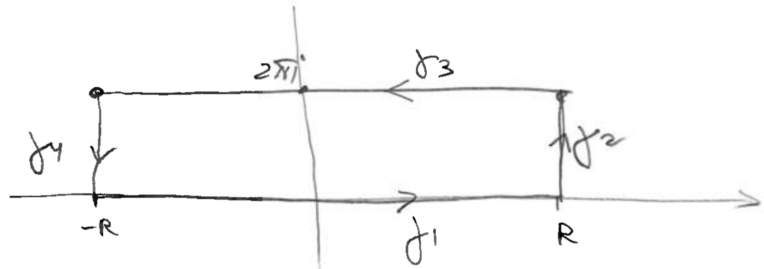
► (Solution of Ex 4)

Key: Close line segment γ_1 from $-R$ to $+R$ by adding line segment γ_2 from R to $R+2\pi i$, then line segment γ_3 from $R+2\pi i$ to $-R+2\pi i$, and finally line segment γ_4 back to $-R$



Cauchy's Residue Thm $\Rightarrow$ $\left(\int_{\gamma_1}+\int_{\gamma_2}+\int_{\gamma_3}+\int_{\gamma_4}\right)\underbrace{\frac{e^{az}}{1+e^z}}_{=:f(z)}dz = 2\pi i\cdot\sum_{z_j\text{-inside}}\mathrm{Res}_{z_j} f(z)$

But out of $\{(2k+1)\pi i \mid k\text{-integer}\}$ we have only $z_1=\pi i$ inside this $\gamma_1+\gamma_2+\gamma_3+\gamma_4$.

and $\mathrm{Res}_{\pi i}\frac{e^{az}}{1+e^z} = \frac{e^{a\cdot\pi i}}{e^{\pi i}} = -e^{a\cdot\pi i}$

Next:

* γ_2 can be parametrized $R+it$ $(0\le t\le 2\pi)$ $\Rightarrow \left|\int_{\gamma_2}\frac{e^{az}}{1+e^z}dz\right| \le 2\pi\cdot\frac{e^{aR}}{e^R-1} \xrightarrow[\text{as } a<1]{R\to+\infty} 0$

* $-\gamma_4$ can be parametrized $-R+it$ $(0\le t\le 2\pi)$ $\Rightarrow \left|\int_{\gamma_4}\frac{e^{az}}{1+e^z}dz\right| \le 2\pi\cdot\frac{e^{-aR}}{1-e^{-R}} \xrightarrow[\text{as } a>0]{R\to+\infty} 0$

! <u>However</u>: $\int_{\gamma_3}\frac{e^{az}}{1+e^z}dz$ <u>does not</u> tend to 0 as $R\to+\infty$ <u>but rather</u> to a multiple of I!

Parametrize $-\gamma_3$: $t+2\pi i$, $-R\le t\le R$, so that

$$\int_{-\gamma_3}\frac{e^{az}}{1+e^z}dz = \int_{-R}^{R}\frac{e^{at}\cdot e^{a\cdot 2\pi i}}{1+e^t}dt \xrightarrow[R\to+\infty]{} e^{a\cdot 2\pi i}\cdot I \Rightarrow \int_{\gamma_3}\frac{e^{az}}{1+e^z}dz \xrightarrow{R\to+\infty} -e^{a\cdot 2\pi i}\cdot I$$

So: $(1-e^{2a\pi i})I = -2\pi i\cdot e^{a\pi i} \Rightarrow I = \frac{2\pi i}{e^{a\pi i}-e^{-a\pi i}} = \frac{2\pi i}{2i\sin(a\pi)} = \frac{\pi}{\sin(a\pi)}$

Thus: $\boxed{I = \pi/\sin(a\pi)}$

Note: In practice, we shall always "close" $[-R,+R]$ via line segments and circle arches.