

Lecture #31

Today: § 6.5. (improper integrals with points of discontinuity)

Setup: Let $a < c < b$ and $f(x)$ be a function continuous on $[a, b] \setminus \{c\}$ but not continuous at c .

$$\left. \begin{aligned} \int_a^c f(x) dx &= \lim_{\epsilon \rightarrow 0^+} \int_a^{c-\epsilon} f(x) dx \\ \int_c^b f(x) dx &= \lim_{\epsilon \rightarrow 0^+} \int_{c+\epsilon}^b f(x) dx \end{aligned} \right\} \begin{aligned} &\implies \int_a^b f(x) dx = \left(\int_a^c + \int_c^b \right) f(x) dx \\ &\text{assuming both exist.} \end{aligned}$$

Example: a) Consider $f(x) = \frac{1}{\sqrt{x}}$ on $[-1, 1]$ with 0 being a value of c above

$$\left. \begin{aligned} \int_{\epsilon}^1 \frac{1}{\sqrt{x}} dx &= \frac{2}{2} x^{\frac{2}{2}} \Big|_{\epsilon}^1 \xrightarrow{\epsilon \rightarrow 0^+} \frac{3}{2} \\ \int_{-1}^{-\epsilon} \frac{1}{\sqrt{x}} dx &= \frac{2}{2} x^{\frac{2}{2}} \Big|_{-1}^{-\epsilon} \xrightarrow{\epsilon \rightarrow 0^+} -\frac{3}{2} \end{aligned} \right\} \implies \int_{-1}^1 \frac{1}{\sqrt{x}} dx = 0$$

b) Consider $f(x) = \frac{1}{x}$ on $[-1, 1]$

$$\int_{\epsilon}^1 \frac{1}{x} dx = \ln x \Big|_{\epsilon}^1 = 1 - \ln \epsilon \xrightarrow{\epsilon \rightarrow 0^+} \infty \implies \text{doesn't exist}$$

likewise $\int_{-1}^{-\epsilon} \frac{1}{x} dx$, $\int_{-1}^1 \frac{1}{x} dx$ don't exist.

To compensate for b) above, one defines principal value

$$\text{p.v. } \int_a^b f(x) dx = \lim_{\epsilon \rightarrow 0^+} \left(\int_a^{c-\epsilon} + \int_{c+\epsilon}^b \right) f(x) dx$$

(when $\int_a^b f(x) dx$ exists, it coincides with p.v. $\int_a^b f(x) dx$).

Example: In part b) above: $\int_{-1}^{-\epsilon} \frac{1}{x} dx + \int_{\epsilon}^1 \frac{1}{x} dx = 0 \quad \forall \epsilon \implies$
 $\implies \text{p.v. } \int_{-1}^1 \frac{1}{x} dx = 0$

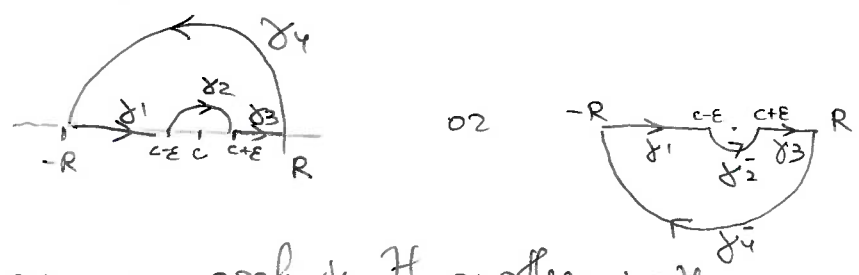
Finally, if $f(x)$ is continuous on $\mathbb{R} \setminus \{c\}$, then we define

Def:
$$\text{p.v. } \int_{-\infty}^{+\infty} f(x) dx = \lim_{\substack{R \rightarrow +\infty \\ \epsilon \rightarrow 0^+}} \left(\int_{-R}^{c-\epsilon} + \int_{c+\epsilon}^R \right) f(x) dx$$

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Following the strategy from the previous 2 lectures, we want to "close up" the segments $[-R, c-\epsilon] \cup [c+\epsilon, R]$ into a closed path not containing singularities on it.

Common choices:



Note: One could also have one arch in \mathbb{H} , another in $-\mathbb{H}$ but this will only complicate computations as then one also needs to compute $\text{Res}_c(\dots)$

Finally, if $f(x)$ is continuous on $\mathbb{R} \setminus \{c_1, \dots, c_k\}$, then

$$\text{p.v.} \int_{-\infty}^{+\infty} f(x) dx = \lim_{\substack{R \rightarrow +\infty \\ \epsilon_1, \dots, \epsilon_k \rightarrow 0^+}} \left(\int_{-R}^{c_1-\epsilon_1} + \int_{c_1+\epsilon_1}^{c_2-\epsilon_2} + \dots + \int_{c_k+\epsilon_k}^{+R} \right) f(z) dz$$

(Note: the radiuses $\epsilon_1, \epsilon_2, \dots, \epsilon_k$ are independent of each other!)

Finally, we shall need:

Lemma 1: If $f(z)$ has a simple pole at $z=c$ and Γ_ϵ is the arch parametrized by $z=c+\epsilon \cdot e^{i\theta}$ with θ from θ_1 to θ_2 , then

$$\lim_{\epsilon \rightarrow 0^+} \int_{\Gamma_\epsilon} f(z) dz = i \cdot (\theta_2 - \theta_1) \cdot \text{Res}_c(f(z))$$

Major application will be to the arches γ_2, γ_2^* as above:

$$\int_{\gamma_2^*} f(z) dz = i \cdot (0 - (-\pi)) \cdot \text{Res}_c f = i\pi \cdot \text{Res}_c(f)$$

$$\int_{\gamma_2} f(z) dz = i \cdot \begin{matrix} (0 - \pi) \\ \uparrow \theta_2 \quad \uparrow \theta_1 \end{matrix} \cdot \text{Res}_c f = -i\pi \cdot \text{Res}_c(f)$$

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Proof of Lemma

$f(z) = \frac{a_{-1}}{z-c} + \underbrace{a_0 + a_1(z-c) + a_2(z-c)^2 + \dots}_{g(z) \text{ - regular at } c}$ - Laurent series expansion in small $D_\epsilon(c)$ (14)

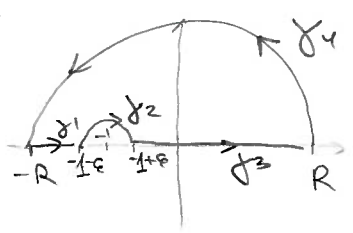
$g(z)$ is there bounded on $D_\epsilon(c)$, i.e. $|g(z)| \leq M \forall z \in D_\epsilon(c)$

$\Rightarrow \left| \int_{\Gamma_\epsilon} g(z) dz \right| \leq M \cdot \text{length}(\Gamma_\epsilon) = M \cdot \epsilon \cdot |\theta_2 - \theta_1| \xrightarrow{\epsilon \rightarrow 0^+} 0$

On the other hand, $\int_{\Gamma_\epsilon} \frac{1}{z-c} dz = \log_{\mathbb{R}}(z-c) \Big|_{z=c+\epsilon e^{i\theta_1}}^{z=c+\epsilon e^{i\theta_2}}$ with \approx not b/w θ_1 & θ_2
 $i(\theta_2 - \theta_1)$ - independent of ϵ

$\underline{\underline{So}}: \int_{\Gamma_\epsilon} f(z) dz \xrightarrow{\epsilon \rightarrow 0^+} i(\theta_2 - \theta_1) a_{-1} = i(\theta_2 - \theta_1) \text{Res}_c(f(z))$

Ex 1: p.v. $\int_{-\infty}^{+\infty} \frac{e^{i \cdot 2x}}{x+1} dx = ?$



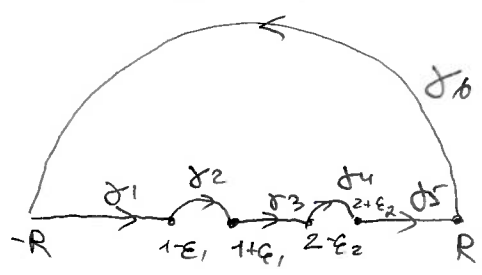
$\left(\int_{\gamma_1} + \int_{\gamma_2} + \int_{\gamma_3} + \int_{\gamma_4} \right) \frac{e^{i \cdot 2z}}{z+1} dz = 0$ as it's analytic inside

But: $\int_{\gamma_4} \frac{e^{i \cdot 2z}}{z+1} dz \xrightarrow{R \rightarrow +\infty} 0$ by Jordan's lemma

$\int_{\gamma_2} \frac{e^{i \cdot 2z}}{z+1} dz \xrightarrow{\epsilon \rightarrow 0^+} i(-\pi) \cdot e^{i \cdot 2(-1)} = -\pi i e^{-2i}$

$\underline{\underline{So}}: \text{p.v.} \int_{-\infty}^{+\infty} \frac{e^{i \cdot 2x}}{x+1} dx = \pi i e^{-2i}$

Ex 2: p.v. $\int_{-\infty}^{+\infty} \frac{e^{ix}}{(x-1)(x-2)} dx = ?$



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$\left(\int_{\gamma_1} + \int_{\gamma_2} + \int_{\gamma_3} + \int_{\gamma_4} + \int_{\gamma_5} + \int_{\gamma_6} \right) \frac{e^{iz} dz}{(z-1)(z-2)} = 0$ by Cauchy's Thm.

But: $\int_{\gamma_6} \frac{e^{iz}}{(z-1)(z-2)} dz \rightarrow 0$ as $R \rightarrow +\infty$

$\int_{\gamma_2} \frac{e^{iz}}{(z-1)(z-2)} dz \rightarrow -i\pi \frac{e^{iz}}{z-2} \Big|_{z=1} = i\pi e^i$ as $\epsilon_1 \rightarrow 0^+$

$\int_{\gamma_4} \frac{e^{iz}}{(z-1)(z-2)} dz \rightarrow -i\pi \frac{e^{iz}}{z-1} \Big|_{z=2} = -i\pi e^{2i}$ as $\epsilon_2 \rightarrow 0^+$

So: p.v. $\int_{-\infty}^{+\infty} \frac{e^{ix}}{(x-1)(x-2)} dx = i\pi(e^i - e^{2i})$

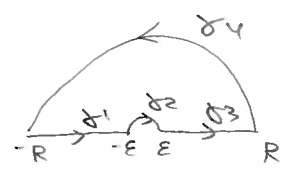
Ex 3: $I = \int_0^{+\infty} \frac{\sin 2x}{x} dx = ?$

First, as in previous lectures, we note that $\frac{\sin(-2x)}{-x} = \frac{\sin 2x}{x}$, hence:

$2I = \text{p.v.} \int_{-\infty}^{+\infty} \frac{\sin 2x}{x} dx$

Second, note that $\sin(2x) = \text{Im}(e^{i \cdot 2x})$

$\Rightarrow I = \frac{1}{2} \text{Im} \left(\text{p.v.} \int_{-\infty}^{+\infty} \frac{e^{i2x}}{x} dx \right)$



$\left(\int_{\gamma_1} + \int_{\gamma_2} + \int_{\gamma_3} + \int_{\gamma_4} \right) \frac{e^{i2z}}{z} dz = 0$

$\int_{\gamma_4} \frac{e^{i2z}}{z} dz \rightarrow 0$ as $R \rightarrow +\infty$ by Jordan's Lemma

$\int_{\gamma_2} \frac{e^{i2z}}{z} dz = -i\pi \cdot e^{i \cdot 2 \cdot 0} = -i\pi$

$\Rightarrow \text{p.v.} \int_{-\infty}^{+\infty} \frac{e^{i2x}}{x} dx = i\pi \Rightarrow \text{Im}(\dots) = \pi \Rightarrow I = \left(\frac{\pi}{2} \right)$

Remark: Another direct way would be to integrate $\frac{e^{i2z}}{z}$ as above and $\frac{e^{i \cdot (-2z)}}{z}$ over $\gamma_1 + \gamma_2 + \gamma_3 + \gamma_4$, and use $\sin 2z = \frac{e^{i2z} - e^{-i2z}}{2i}$