Lecture #32

* Key estimates from last 3 classes

1) $\displaystyle\int_{\gamma} \frac{P(z)}{Q(z)} \xrightarrow[R\to+\infty]{} 0$ if P, Q - polynomials with $\deg Q \geq \deg P + 2$

γ = circle $|z| = R$ or any part of it

2a) $\displaystyle\int_{\gamma_R^+} \frac{P(z)}{Q(z)} e^{imz} \xrightarrow[R\to+\infty]{} 0$ if $m \in \mathbb{R}_{>0}$ and P, Q - polynomials, $\deg Q \geq \deg P + 1$

2b) $\displaystyle\int_{\gamma_R^-} \frac{P(z)}{Q(z)} e^{imz} \xrightarrow[R\to+\infty]{} 0$ if $m \in \mathbb{R}_{<0}$ and P, Q - polynomials, $\deg Q \geq \deg P + 1$

} Jordan lemma

3a) $\displaystyle\int f(z)dz \xrightarrow{\varepsilon \to 0^+} -\pi i \operatorname{Res}_c f(z)$ if c - simple pole of $f(z)$

= top-half of radius ε circle oriented clockwise

3b) $\displaystyle\int f(z)dz \xrightarrow{\varepsilon \to 0^+} \pi i \operatorname{Res}_c f(z)$ if c - simple pole of $f(z)$

= bottom-half of radius ε circle oriented counter clockwise

} Lecture #31

Today: $\boxed{\text{§ 6.6 = Integrals of multivalued functions}}$

This is the last and definitely the hardest type of integrals we encountered so far. Let's illustrate the idea with our first exercise.

Ex 1: Evaluate $I = \displaystyle\int_0^{+\infty} \frac{1}{\sqrt{x}(x+1)} dx$

Note: 1) The integrand is not even, so we can't use our previous trick of saying it's $\frac{1}{2} pv \displaystyle\int_{-\infty}^{+\infty} \dots$

2) $\frac{1}{\sqrt{x}}$ has a singularity at 0

3) Most importantly, while $\sqrt{x}$ is well-defined on $[0, \infty)$, extending it to $\mathbb{C}$ gives a multivalued function $\sqrt{z}$!

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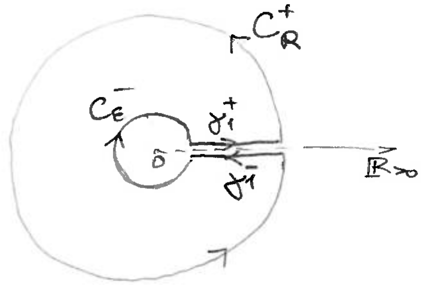
- We shall evaluate I by integrating $f(z) = \frac{1}{z+1} e^{-\frac{1}{2}\log_0 z}$ over the contour

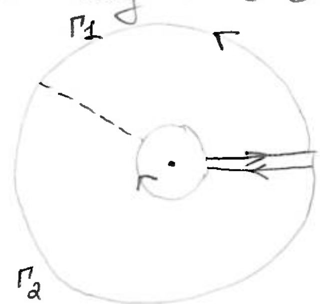


Here γ_1^+ and γ_1^- denote the same line segment $[\varepsilon, R]$, but the meaning of $f(z)|_{\gamma_1^\pm}$ is taking limit of $f(z)$ as we approach from top half or bottom half of the $\mathbb{C}$-plane.

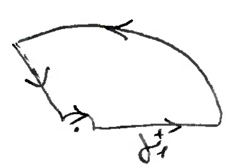
Here: $f(z) \longrightarrow \frac{1}{\sqrt{x}\,(x+1)}$ as $z \to x \in \mathbb{R}_{>0}$ in $\mathcal{H} = \{\text{upper half-plane}\}$

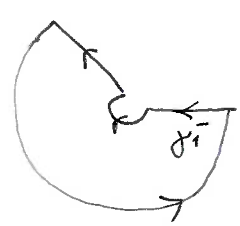
$f(z) \longrightarrow -\frac{1}{\sqrt{x}\,(x+1)}$ as $z \to x \in \mathbb{R}_{>0}$ in $(-\mathcal{H}) = \{\text{lower half-plane}\}$

Warning: While the above contour is not exactly the one to which Cauchy Residue theorem applies on the nose, we can split it along the dashed arrow



to get two closed positively oriented contours Γ_1 & Γ_2.

- On $\Gamma_1 =$  we note that $\log_0 z$ is really the same as $\log_{-\delta'}(z)$ for some small $\delta' > 0$ (where $\log_0 z|_{\gamma_1^+}$ is understood in terms of limit as above)

- On $\Gamma_2 =$  we note that $\log_0 z$ is really the same as $\log_{\delta'}(z)$ for small $\delta' > 0$

Applying Cauchy Residue theorem to Γ_1 & Γ_2 and adding results will now give us the Residue theorem in the present setup.

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(Continuation of Solution of Ex1)

Step 1 : Residue thm (explained on the previous page)

$$\left.\begin{array}{l}\oint_{\Gamma} f(z)dz = 2\pi i \, Res_{-1} f(z) \\ f(z) = \dfrac{e^{-\frac{1}{2}\log_0 z}}{z+1} \Rightarrow Res_{-1} f(z) = e^{-\frac{1}{2}\log_0(-1)} = e^{-\frac{1}{2}(0+i\pi)} = -i\end{array}\right\} \Rightarrow \oint_{\Gamma} f(z)dz = 2\pi.$$

Step 2 : Estimates on "big" & "small" circles.

$$\left|\int_{C_R^+} f(z)dz\right| \leq 2\pi R \cdot \frac{1}{R^{1/2}(R-1)} \xrightarrow{R\to+\infty} 0$$

$$\left|\int_{C_\varepsilon^-} f(z)dz\right| \leq 2\pi\varepsilon \cdot \frac{1}{\varepsilon^{1/2}(1-\varepsilon)} \xrightarrow{\varepsilon\to 0^+} 0$$

Step 3 : Relating $\int_{\gamma_1^-}$ to $\int_{\gamma_1^+}$

As follows from previous page: $\int_{-\gamma_1^-} f(z)dz = -\int_{\gamma_1^+} f(z)dz \Rightarrow \int_{\gamma_1^-} f(z)dz = \int_{\gamma_1^+} f(z)dz$

Step 4 : Deducing I

Taking limit $R\to+\infty, \varepsilon\to 0^+$ in Step 1 and using Steps 2-3, we get

$$2I = 2\pi \Rightarrow I = \pi$$

Note: The reason why we choose the branch $\log_0 z$ is precisely so that it doesn't interfere with "extra big & small circles". Note, however, that if we had $\frac{1}{x}$ instead of $\frac{1}{\sqrt{x}}$ so that there wouldn't be discontinuity on $\mathbb{R}_{>0}$ as above, then $\int_{\gamma_1^+ + \gamma_1^-}$ would be 0!

Our next exercise is a combination of above with Lecture 31!

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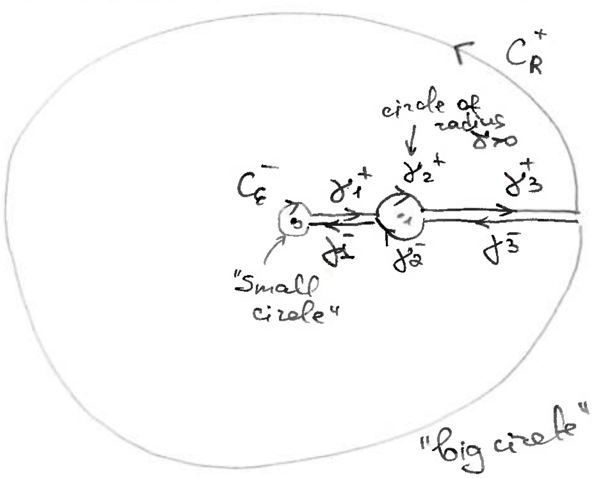
Ex2: For $0<\lambda<1$, evaluate $I = \text{p.v.} \int_0^{+\infty} \frac{1}{x^\lambda (x-1)}\,dx$

"p.v" is due to singularity at $1 \in (0,+\infty)$

▶ Our integrand will be similar to Ex 1:

$$f(z) = \frac{1}{(z-1)e^{\lambda \log_0 z}}$$

while the contour will now be:



$$\Gamma = \gamma_1^+ + \gamma_2^+ + \gamma_3^+ + C_R^+ + \gamma_3^- + \gamma_2^- + \gamma_1^- + C_\varepsilon^-$$

Step 1: $\oint_\Gamma f(z)\,dz = 0$

Step 2: Estimates for $\int_{C_R^+}$ & $\int_{C_\varepsilon^-}$

$$\left|\int_{C_R^+} f(z)\,dz\right| \le 2\pi R \cdot \frac{1}{R^\lambda (R-1)} \xrightarrow[R\to+\infty]{} 0 \quad \text{as } \lambda>0$$

$$\left|\int_{C_\varepsilon^-} f(z)\,dz\right| \le 2\pi \varepsilon \cdot \frac{1}{\varepsilon^\lambda (1-\varepsilon)} \xrightarrow[\varepsilon\to 0^+]{} 0 \quad \text{as } \lambda<1$$

Step 3: $\gamma_1^\pm, \gamma_3^\pm$ - contribution

Clearly: $\lim\limits_{\substack{R\to+\infty \\ \varepsilon\to 0^+ \\ \delta\to 0^+}} \int_{\gamma_1^+ + \gamma_3^+} f(z)\,dz = I$

Also: $\lim\limits_{\substack{R\to+\infty \\ \varepsilon\to 0^+ \\ \delta\to 0^+}} \int_{\gamma_1^- + \gamma_3^-} f(z)\,dz = -e^{-2\pi i\lambda} \cdot I$ as $\lim\limits_{z\to x\in \mathbb{R}_{>0} \text{ in } (-\mathcal{H})} \frac{1}{(z-1)e^{\lambda \log_0 z}} = e^{-\lambda\cdot 2\pi i} \cdot \frac{1}{(x-1)x^\lambda}$

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▸ (Continuation)

Step 4: Treatment of $\int_{\gamma_2^{\pm}}$

On γ_2^+: $\displaystyle\int_{\gamma_2^+} f(z)dz \xrightarrow[\delta\to 0^+]{} -i\pi \operatorname{Res}_1 f(z) = -i\pi \cdot \frac{1}{e^{\lambda \log_0(1)}} = -i\pi$

but on γ_2^- we need to account for the same $e^{-\lambda\cdot 2\pi i}$ as in Step 3:

$$\int_{\gamma_2^-} f(z)dz \xrightarrow[\delta\to 0^+]{} -i\pi\cdot e^{-2\pi i\lambda}$$

Step 5: Deducing I

Taking the limit $R\to+\infty$, $\varepsilon\to 0^+$, $\delta\to 0^+$ of Step 1 and using Steps 2-4 gives us:

$$(1-e^{-2\pi i\lambda})I = i\pi(1+e^{-2\pi i\lambda})$$

$$\Downarrow$$

$$I = i\pi\cdot\frac{e^{i\cdot\pi\lambda}+e^{-i\pi\lambda}}{e^{i\cdot\pi\lambda}-e^{-i\pi\lambda}} = \pi\cot(\pi\lambda)$$

So: $I = \pi\cot(\pi\lambda)$

[**Note**: The answer above must be real, so if you got smth complex you must simplify to real number]

However, the same technique may be also used even when there is no multi-valued function at the first glance, such as:

Ex 3: Find $I = \displaystyle\int_0^{+\infty}\frac{dx}{(x+1)(x^2+2x+2)}$

↑ textbook, p. 353, Example 4.