Lecture #33 §6.7: Argument Principle and Rouché's theorem

Def.: Function $f(z)$ is *meromorphic* at domain D if at every point $z \in D$ it's either analytic or has a pole

Today: $\begin{cases} D = \text{inside of some simple closed curve } \gamma \\ f\text{- analytic and nonzero on } \gamma \text{, and meromorphic on } D. \end{cases}$

Claim: In this setup, f has a finite number of zeroes & poles in D

(discuss in class! key: Bolzano-Weierstrass theorem)

Let's introduce

$$\begin{cases} N_0(f) = \text{sum of orders of all zeroes of } f(z) \text{ inside } D \\ N_p(f) = \text{sum of orders of all poles of } f(z) \text{ inside } D \end{cases}$$

Our first main result today is going to express $\oint_\gamma \frac{f'(z)}{f(z)}dz$ via $N_0(f)$, $N_p(f)$.

First, let's look at the "toy example".

Ex 1: Evaluate $\oint_{|z|=3} \frac{f'(z)}{f(z)}dz$ for $f(z) = \frac{(z-1)^2(z-2)^3(z-5)^{10}}{(z^2+1)^3(z^2+4)^5}$

Calculus $\Rightarrow$ $\frac{f'(z)}{f(z)} = \frac{2}{z-1} + \frac{3}{z-2} + \frac{10}{z-5} - \frac{3}{z-i} - \frac{3}{z+i} - \frac{5}{z-2i} - \frac{5}{z+2i}$

As $\oint_{|z|=3} \frac{dz}{z-c} = \begin{cases} 2\pi i & \text{if } |c|<3 \\ 0 & \text{if } |c|>3 \end{cases}$ $\Rightarrow$ $\oint_{|z|=3} \frac{f'(z)}{f(z)}dz = 2\pi i \left(\underbrace{(2+3)}_{N_0(f)} - \underbrace{(3+3+5+5)}_{N_p(f)}\right) = -22\pi i$ ▫

Now we are ready to state the first theorem for today:

Theorem 1 ("Argument Principle"): If f is analytic and nonzero at each point of a simple closed positively oriented contour Γ and meromorphic inside Γ, then

$$\boxed{\oint_\Gamma \frac{f'(z)}{f(z)}dz = 2\pi i \left(N_0(f) - N_p(f)\right)}$$

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Proof of Theorem 1

- Singular points of $\frac{f'(z)}{f(z)}$ are either zeroes or poles of $f(z)$

Case 1: If z_0 - zero of $f(z)$ inside Γ, then looking at Taylor expansion of $f(z)$ around z_0, we get $f(z) = (z-z_0)^n \underbrace{(\overset{\neq 0}{a_n} + a_{n+1}(z-z_0) + \dots)}_{\tilde{f}(z)}$

where n = order of zero at z_0

Note: $\tilde{f}(z)$ analytic in nbhd of z_0 and $\tilde{f}(z_0) = a_n \neq 0$

Hence: $$\frac{f'(z)}{f(z)} = \frac{n(z-z_0)^{n-1}\cdot\tilde{f}(z) + (z-z_0)^n\tilde{f}'(z)}{(z-z_0)^n\tilde{f}(z)} = \frac{n}{z-z_0} + \frac{\tilde{f}'(z)}{\tilde{f}(z)}$$

As $\frac{\tilde{f}'(z)}{\tilde{f}(z)}$ is analytic in $D_\varepsilon(z_0)$ for some $\varepsilon > 0$, we get

$$\boxed{\oint_{C_\varepsilon(z_0)} \frac{f'(z)}{f(z)}dz = n\cdot 2\pi i}$$

Case 2: If z_0 - pole of $f(z)$ inside Γ, then looking at the Laurent series expansion in small punctured disk centered at z_0, we get

$f(z) = (z-z_0)^{-n}\cdot \underbrace{(\overset{\neq 0}{a_{-n}} + a_{1-n}(z-z_0) + \dots)}_{\tilde{f}(z)\text{ - analytic at nbhd of } z_0}$ with n = order of pole at z_0

Arguing as above, we then get $\boxed{\oint_{C_\varepsilon(z_0)} \frac{f'(z)}{f(z)}dz = 2\pi i\cdot(-n)}$

Now: As $\frac{f'(z)}{f(z)}$ is analytic inside $\Gamma \setminus \{\text{zeros \& poles}\}$, we get

$$\oint_\Gamma \frac{f'(z)}{f(z)}dz = \sum_{z_0\text{ - zero or pole}} \oint_{C_\varepsilon(z_0)} \frac{f'(z)}{f(z)}dz = 2\pi i\,(N_0(f) - N_p(f))$$

∎

Let's now get a more geometric interpretation of above integral.

Recall (from calculus): If $f: [a,b] \to \mathbb{R}_{>0}$ then $\boxed{\frac{f'(x)}{f(x)} = \frac{d}{dx}\ln(f(x))}$

"logarithmic derivative"

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Now, in the complex setup, we need to choose a branch of log which may not be possible for all $\{f(z) \mid z \in \gamma\}$. However, we can clearly break $\gamma = \gamma_1 + \gamma_2 + \dots + \gamma_N$ into several subarcs $\{\gamma_j\}_{j=1}^{N}$ (say between points y_j & y_{j+1} on γ) so that the argument of $f(z)$ varies by less than 2π for $z \in \gamma_j$ which then allows to pick a branch of log (separately for each γ_j). By Fundamental theorem of calculus:

$$\int_{\gamma_j} \frac{f'(z)}{f(z)}\,dz = \left\{ \ln|f(z)| + i \arg_{\tau_j} f(z) \right\} \Big|_{y_j}^{y_{j+1}} \quad \text{with } \tau_j \text{ determining the branch}$$

$$\Downarrow$$

$$\underbrace{\oint_{\gamma} \frac{f'(z)}{f(z)}\,dz}_{} = (\ln|f(y_2)| - \ln|f(y_1)|) + (\ln|f(y_3)| - \ln|f(y_2)|) + \dots + (\ln|f(y_1)| - \ln|f(y_N)|)$$

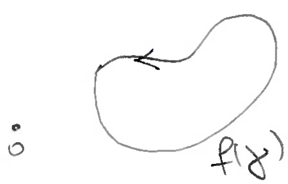
$$+ i \cdot \Delta_\gamma \arg f(z) = \boxed{i \cdot \underbrace{\Delta_\gamma \arg f(z)}_{\text{"net excursion" of } \arg f(z)}}$$

- (key: similarly defined $\Delta_\gamma \ln|f(z)|$ is clearly zero as seen above!)

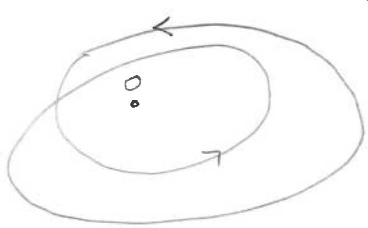
Combining with Theorem 1, we get:

$$\boxed{\text{Corollary: } \Delta_\gamma \arg f(z) = 2\pi (N_0(f) - N_p(f))}$$

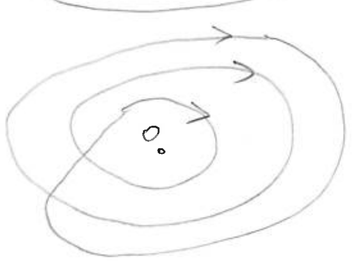
Examples:



$\Delta_\gamma \arg f(z) = 0$



$\Delta_\gamma \arg f(z) = 4\pi \quad \Rightarrow N_0(f) \geq 2$



$\Delta_\gamma \arg f(z) = -6\pi \quad \Rightarrow N_p(f) \geq 3$

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Theorem 2 (Rouché's Theorem): If γ - simple closed curve and f, h are analytic on γ and inside it, and $|h(z)| < |f(z)|$ for any $z \in \gamma$, then:

$$N_0(f) = N_0(f+h)$$

i.e. f and $f+h$ have the same number of zeroes inside γ

- "Pictorial proof" (see pages 360-361 of the textbook): think of $f(z)$ as being the path of you and $f(z)+h(z)$ - path of your dog (with $z \in \gamma$) as you walk around the lamppost located at the origin. The condition $|h(z)| < |f(z)|$ $\forall z \in \gamma$ guarantees that the leash never extends past the lamppost. Hence:

$$\underbrace{\Delta_\gamma \arg(f(z))}_{\substack{\| \\ 2\pi \cdot N_0(f)}} = \underbrace{\Delta_\gamma \arg(f(z)+h(z))}_{\substack{\| \\ 2\pi \cdot N_0(f+h)}}$$

$\longleftarrow$ by Corollary $\longrightarrow$

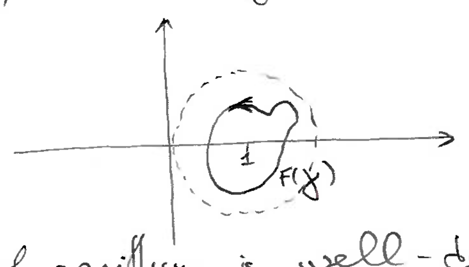
$$\Downarrow$$

$$\boxed{N_0(f) = N_0(f+h)}$$

- Rigorous Proof of Theorem 2

Let $g = f+h$. Then $|h| < |f|$ on $\gamma \Rightarrow \left|1 - \frac{g}{f}\right| < 1$ on γ.

Consider $F(z) = \frac{g(z)}{f(z)}$, so that $F(\gamma)$ is inside $D_1(1)$:



Then the principal logarithm is well-defined on $F(\gamma)$

So: $\frac{d}{dz} \operatorname{Log} F(z) = \frac{F'(z)}{F(z)} = \frac{g'(z) \cdot f(z) - g(z) f'(z)}{f(z)^2} \cdot \frac{f(z)}{g(z)} = \frac{g'(z)}{g(z)} - \frac{f'(z)}{f(z)}$

Thus: $0 = \oint_\gamma \frac{d}{dz} \operatorname{Log} F(z)\, dz = \oint_\gamma \frac{F'(z)}{F(z)} dz = \oint_\gamma \frac{g'(z)}{g(z)} dz - \oint_\gamma \frac{f'(z)}{f(z)} dz$

$\|$ Argument Principle

$2\pi i \,(N_0(g) - N_0(f))$

This proves $\underline{N_0(f) = N_0(g)}$ $\quad g = f+h$

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The key application of Rouché's theorem is to the location of zeroes of analytic functions $g(z)$ by comparing to ~~zeroes~~ of simpler function $f(z)$

Example: Show that all roots of $g(z)=z^4+7z^2+3z+5$ lie in disk $|z|<3$

▸ We want to pick $f(z)$ to be the "leading term" for z on circle of radius 3.

Clear: $|z^4|=81$, $|z^2|=9$, $|z|=3$ on this circle.

So: let's pick $f(z)=z^4$ and $h(z)=g(z)-f(z)=7z^2+3z+5$.

Then: $|h(z)|\le 7\cdot 9+3\cdot 3+5=77<81=|f(z)|$ for all z on this circle

$$\Downarrow$$

$$N_0(g)=N_0(f)$$

But clearly z^4 has only one zero at $z_0=0$ of order 4 $\Big\} \Rightarrow N_0(g)=4$

Since any polynomial of degree 4 has 4 roots $\Rightarrow$ all inside $D_3(0)$ ∎

Ex2: How many zeroes does $g(z)=z^3+8z+25$ have on $D_2(0)=\{z \mid |z|<2\}$?

▸ For $|z|=2$ have $|z^3|=8$, $|z|=2 \Rightarrow |z^3+8z|\le 24$

Thus: we should take $f(z)=25$ and $h(z)=g(z)-f(z)=z^3+8z$.

Then, Rouché's Thm $\Rightarrow N_0(g)=N_0(f)$. But f is a nonzero constant, hence, has no zeroes!

So: $g(z)$ has no zeroes on $D_2(0)$! ∎

Ex3: Show that all zeroes of $g(z)=z^5-6z^2+7.5$ fall in the annulus $\{1<|z|<2\}$.

▸ This is a combination of above Example + Ex2.

- For $\gamma=\{|z|=2\}$, take $f(z)=z^5$, $h(z)=-6z^2+7.5$. Then $|h(z)|\le 31.5<|f(z)|$ for $|z|=2 \Rightarrow N_0(g)=N_0(f)=5 \overset{g\text{-polynomial of deg }5}{\Longrightarrow}$ all zeroes of $g(z)$ are in $D_2(0)$.

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(Continuation)

• For $\gamma = \{|z|=1\}$, we take $f(z) = 7.5$ and $h(z) = z^5 - 6z^2$.

Then $|h(z)| \le 7 < |f(z)|$ for all $|z|=1$ $\Rightarrow N_0(g) = N_0(f) = 0 \Rightarrow$ no zeroes in $D_1(0)$.

But above inequalities also imply no zeroes on $C_1(0)$.

Combining these two steps, we get the result! ∎

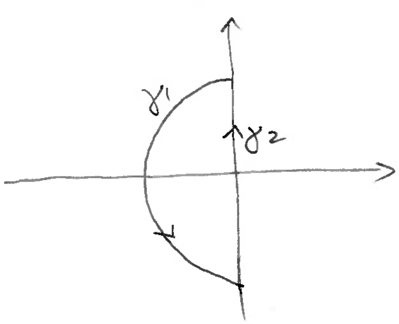
Ex 4: Prove that $z+4+3e^{2z} =: g(z)$ has precisely one zero in $D = \{z \in \mathbb{C} : \operatorname{Re} z \le 0\}$

! Warning: First, we note that $g(z)$ is not a polynomial (as in previous Ex)

More importantly D is not an interior of some closed curve.

▸ Note that $|e^{2z}| \le 1$ for $z \in D$

For any $\rho > 7$, consider the half circle in D of radius ρ centered at the origin, followed up by vertical segment from $-\rho i$ to ρi.



$\gamma = \gamma_1 + \gamma_2$

Then: $|z+4| \ge 4$ on γ_2

$|z+4| \ge \rho - 4 > 3$ on γ_1 b/c $\rho > 7$

Take $f(z) = z+4$, $h(z) = 3e^{2z}$, so that $|h(z)| < |f(z)|$ for all $z \in \gamma$.

By Rouché's theorem: $N_0(g) = N_0(f) = 1$ ← single simple zero at $z_0 = -4$

Thus, $g(z)$ has a single zero inside above γ for any $\rho > 7$. Taking limit $\rho \to +\infty$, we clearly get the claim ∎

↑ indeed, if there were at least two zeroes $z_1, z_2 \in D$, then for $\rho > 7, |z_1|, |z_2|$ the above argument yields a contradiction!

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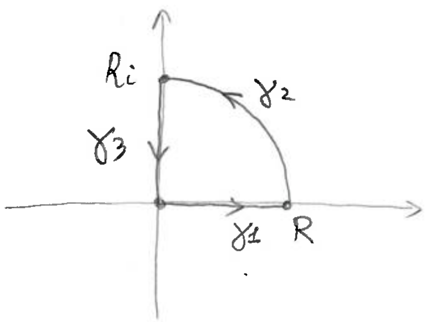
We conclude with the following considerably harder example.

Example: Show that the polynomial $g(z) = z^3 - 2z^2 + 4$ has exactly one simple root in the first quadrant $\{z \in \mathbb{C} : \operatorname{Re} z \geq 0, \operatorname{Im} z \geq 0\}$.



By the argument principle applied to the contour $\Gamma = \gamma_1 + \gamma_2 + \gamma_3$ depicted to the left (with $R > 0$ - big real) it suffices to evaluate $\Delta_\Gamma \arg g(z)$

- γ_1-part: parametrized $0 \leq t \leq R$

$$\left.\begin{array}{l} g(0) = 4 \\ g'(t) = 3t^2 - 4t = t(3t-4) \Rightarrow g(t) \downarrow \text{ on } [0, 4/3] \text{ and } \uparrow \text{ on } [4/3, +\infty) \\ g(\frac{4}{3}) = \frac{64}{27} - 2 \cdot \frac{16}{9} + 4 > 0 \end{array}\right\} \Rightarrow$$

$\Rightarrow$ g-image of γ_1 is a path on $\mathbb{R}_{>0}$ starting at 4, then decreasing till $g(4/3)$ and then increasing up to $g(R) \sim R^3$.

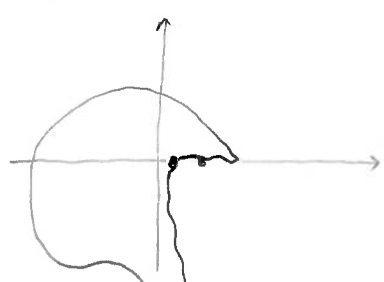
- γ_2-part: parametrized $Re^{i\theta}$ with $0 \leq \theta \leq \pi/2$

As $R \gg 1$, it roughly looks as the arch of circle of radius R^3 with total angle $\sim 3\pi/2$

- γ_3-part: parametrize $-\gamma_3$ by $i \cdot t$, $0 \leq t \leq R$

$$g(it) = -it^3 + (4 + 2t^2) = \underbrace{(4+2t^2)}_{>0} + \underbrace{(-t^3)}_{<0} \cdot i$$

So: the overall image of Γ under the map $z \mapsto g(z)$ roughly looks as:



$\Rightarrow \Delta_\Gamma \arg g(z) = 2\pi \Rightarrow \boxed{N_0(g) = 1}$

Taking limit $R \to +\infty$, we get answer: $\boxed{1}$