Lecture #36

* Last time (before exam): Argument principle & Rouché's theorem.
  → go over the proof of Rouché's thm (see p. 4 of Notes for Lecture 33)

Note: Applying the same proof in the case when h - analytic on & inside γ, while f - analytic on γ and meromorphic inside γ, with still $|h(z)|<|f(z)|$ for all $z\in\gamma$, we get

$$N_0(f)-N_p(f)=N_0(f+h)-N_p(f+h)$$

But if z_0 - pole of f of order m, then z_0 - also order m pole of $f+h$ (inside γ)

$\Rightarrow N_p(f)=N_p(f+h)$. So:

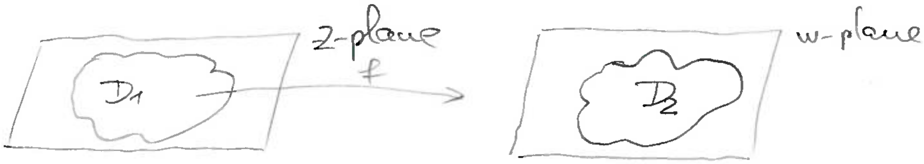
$\boxed{\text{Claim: In the above setup, we again have } N_0(f)=N_0(f+h)}$

↑ see Exercises 14-15 to §6.7 (p. 365).

Rmk: Rouché's theorem also provides another proof of the Fund. Thm. Algebra. Given any polynomial $p(z)=a_0+a_1z+\_+a_{n-1}z^{n-1}+a_nz^n$ with $a_n\neq 0$ let's pick $f(z)=a_nz^n$ and $h(z)=a_0+a_1z+\_+a_{n-1}z^{n-1}$. Then, as $\lim\limits_{z\to\infty}\left(\frac{a_{n-1}}{z}+\_+\frac{a_1}{z^{n-1}}+\frac{a_0}{z^n}\right)=0$, we see $\exists R>0$ s.t. $|h(z)|<|f(z)|$ for all $z\in C_R(0)=\{z: |z|=R\}$. Then: $N_0(p)=N_0(f)=n$

Finally, §6.7 ended with the "Open Mapping" property of non-constant analytic functions, but we'll get to it at the end of today's class.

* Today: Geometric Properties of $z\mapsto w=f(z)$; $\boxed{\S 7.1-7.2}$



Identifying $z\in\mathbb{C}$ with $(x,y)\in\mathbb{R}^2$, can think of f as $(u(x,y), v(x,y))$

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Theorem 1: If $f: D_1 \to D_2$ is analytic 1-to-1 and $f'(z) \neq 0$ $\forall z \in D_1$, then the inverse $f^{-1}: D_2 \to D_1$ is analytic and $\frac{df^{-1}}{dw}(w) = \frac{1}{\frac{df}{dz}(z)}$ with $w = f(z)$

▶ This was proved in [Lecture 11, Theorem 3] and is crucially based on the inverse function theorem and $\det\begin{pmatrix} u_x & u_y \\ v_x & v_y \end{pmatrix} \underset{\text{Cauchy-Riemann}}{=} |f'(z)|^2 \neq 0$ ■

Theorem 2: In the above setup, if $\phi = \phi(u,v)$ is harmonic on D_2, then $\phi \circ f$ - harmonic on D_1

▶ ϕ is harmonic at $w_0 = f(z_0)$ $\Rightarrow$ on some small open disk $D_r(w_0) \subset D_2$ there is an analytic function $g: D_r(w_0) \to \mathbb{C}$ s.t. $\phi = \mathrm{Re}(g)$. But then $\phi \circ f = \mathrm{Re}(g \circ f)$ which holds true on some small disk $D_\delta(z_0) \subseteq D_1$ (which exists as f-continuous $\Rightarrow \exists \delta > 0$ with $D_\delta(z_0) \subseteq f^{-1}(D_r(w_0))$)

But f, g-analytic $\Rightarrow g \circ f$-analytic $\Rightarrow \mathrm{Re}(g \circ f)$ - harmonic ■

Application: Solving Dirichlet problem $\nabla^2 \phi = 0$ in domain D (with specified boundary value) by constructing analytic 1-to-1 map $D \xrightarrow{f} \tilde{D}$, picking harmonic $\tilde{\phi}$ on $\tilde{D}$, and then constructing $\phi = \tilde{\phi} \circ f$

Recall: Know explicit solutions for:
- washers, wedges, walls
- $\underbrace{\text{disks (Poisson's integral f-la)}}_{\text{Lecture 18}}$ & $\underbrace{H = \text{upper half plane}}_{\text{this was on Homework 7}}$

Lecture #36 §7.2

* "Local" properties

Theorem 3: If f-analytic at z_0 and $f'(z_0)\neq 0$, then $\exists \varepsilon>0$ s.t. f is 1-to-1 on $D_\varepsilon(z_0)$

► One way: same as for Theorem 1 - just use inverse function theorem

Other proof - see book, p. 378: Pick $\varepsilon>0$ s.t. $|f'(z)-f'(z_0)|<\frac{1}{2}|f'(z_0)|$ for any $z\in D_\varepsilon(z_0)$ (here use f'-analytic $\Rightarrow$ continuous). For any two $z_1, z_2 \in D_\varepsilon(z_0)$, let γ = the segment from z_1 to z_2, so that:

$$|f(z_1)-f(z_2)| = \left|\int_\gamma f'(z)dz\right| = \left|\underbrace{\int_\gamma f'(z_0)dz}_{=f'(z_0)\cdot(z_2-z_1)} + \underbrace{\int_\gamma (f'(z)-f'(z_0))dz}_{|\dots|\leq \frac{1}{2}|f'(z_0)|\cdot|z_2-z_1|}\right| \geq \underbrace{\tfrac{1}{2}|f'(z_0)|\cdot|z_2-z_1|}_{>0}$$

Non-example: For any $n\geq 2$-integer, the map $z\to z^n$ is actually n-to-1 in any $D_\varepsilon(0)\setminus\{0\}$.

Def: A map $f: D_1 \to D_2$ is called conformal at point $z_0\in D_1$ if it preserves angles b/w curves intersecting at z_0.

Theorem 4: If f-analytic and $f'(z_0)\neq 0$, then f - conformal at z_0

Non-example: For any $n\geq 2$-integer, the map $z\mapsto z^n$ at $z_0=0$ multiplies angles by n! (so is not conformal)

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Proof of Thm 4

- Pick any curve γ passing through z_0 and parametrize it by $z(t)$, so that $z_0 = z(t_0)$

Then, the tangent vector (a.k.a. velocity vector) is $\vec{T} = z'(t_0)$

Now, the curve $f(\gamma)$ is naturally parametrized via $f(z(t))$, and so the corresponding tangent vector $\vec{\tilde{T}}$ at $w_0 = f(z_0)$ is

$$\frac{d}{dt}\left(f(z(t))\right) = f'(z(t_0))\cdot z'(t_0) = f'(z_0)\cdot z'(t_0)$$

As $f'(z_0)\neq 0$, let's write in polar coordinates $f'(z_0) = re^{i\varphi}$. Then, we see that $\vec{\tilde{T}}$ is obtained from $\vec{T}$ by rotating by angle φ and scaling by r

Thus: the angle between any two curves passing through z_0 is preserved ∎

* "Global" properties

Theorem 5 ("Open Mapping property"): Any non-constant analytic function maps open sets to open sets

- Pick any z_0 and first assume that $f(z_0)=0$. As f - nonconstant, $\exists \varepsilon > 0$ such that f has no zeroes in $|z-z_0|\leq \varepsilon$ (as zeroes - isolated set).

Let $c := \min\limits_{|z-z_0|=\varepsilon} |f(z)| > 0$. For any $\delta\in\mathbb{C}$ with $|\delta|<c$, apply Rouche's theorem to $\gamma = C_\varepsilon(z_0)$, $f(z)$, $h(z)\equiv -\delta$ to deduce $N_0(f-\delta) = N_0(f) > 0$

Hence: f takes value δ inside $C_\varepsilon(z_0) \Rightarrow f(D_\varepsilon(z_0))$ contains $D_c(0)$.

Now, if $f(z_0) = w_0 \neq 0$, then just apply above argument to $\tilde{f}(z) = f(z) - w_0$. ∎

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The last theorem for today is:

Theorem 6 (Riemann Mapping Theorem):

a) Let D be any simply connected domain in $\mathbb{C}$, but not entire $\mathbb{C}$. Then there is 1-to-1 analytic function f mapping D onto $\underbrace{D_1(0)}_{\text{unit disk}}$

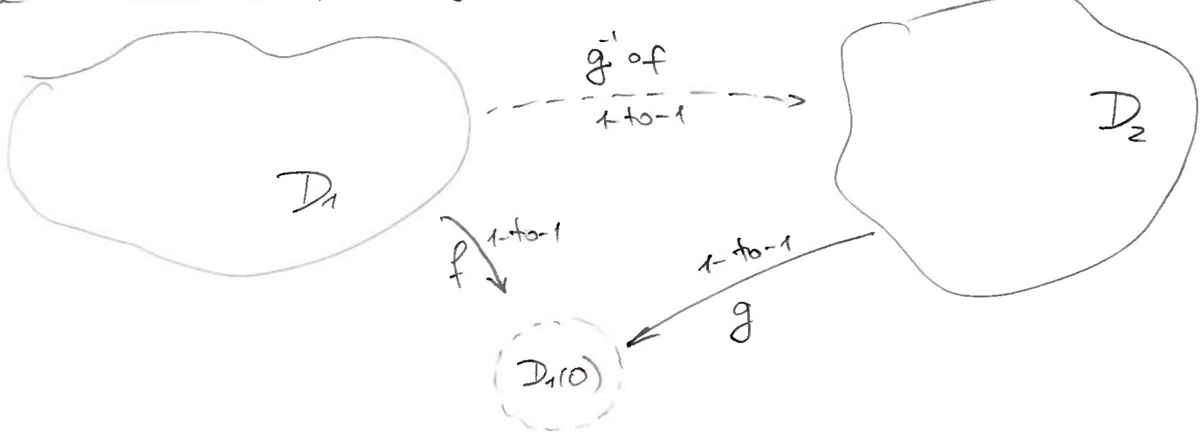
b) For any $z_0 \in D$ and direction vector through z_0, the map f in a) is unique such that $f(z_0) = 0 \in D_1(0)$ and f maps direction vector to the direction of positive real axis

↑ we shall not prove this harder result.

Note: a) f is quite implicit (no construction for it)

b) data in b)-part gives "3 degrees of freedom".

In particular, given two simply connected domains D_1, D_2 in $\mathbb{C}$ which are not all $\mathbb{C}$, we get that there is 1-to-1 analytic map b/w them:



Next lectures: some explicit examples of such f. (§7.3–7.4)

Remark: Note that Theorem 6 clearly fails for $D = \mathbb{C}$, due to Liouville's theorem.