Lecture #37

* Last time: geometric properties of analytic functions.

Ex1 (True/False): Any nonconstant analytic function maps domains to domains?

▸ Recall that $D \subseteq \mathbb{C}$ is a domain if D-open & connected.

The fact that $f(D)$-open for any analytic function D is due to the Open Mapping property (Theorem 5 of Lecture 36).

Now if $z_1, z_2 \in D$, then there is a continuous curve γ from z_1 to z_2, lying solely in D, hence, $f(z_1)$ & $f(z_2)$ are connected by a continuous curve $f(\gamma)$ lying in $f(D)$. This proves that $f(D)$- connected.

So: $f(D)$ - domain $\Rightarrow$ True! ∎

* Today: Some of the basic conformal transformations $\boxed{\S 7.3}$

- translation: $z \mapsto f(z) = z + c$ with $c \in \mathbb{C}$ — preserves angles & lengths
- magnification: $z \mapsto f(z) = \rho \cdot z$ with $\rho \in \mathbb{R}_{>0}$ — preserves angles but changes lengths in ρ times
- rotation (around origin): $z \mapsto f(z) = e^{i\varphi} \cdot z$ with $\varphi \in \mathbb{R}$ — preserves angles & lengths

Composing these basic transformations we get:

Def: Linear transformation is a map $z \mapsto f(z) = az + b$ for any $a, b \in \mathbb{C}$, $a \neq 0$

Note: • the inverse of $z \mapsto w = az + b$ is $w \mapsto a^{-1} w - \frac{b}{a}$ - also linear

• composition of linear transformations $z \mapsto az + b = w$ and $w \mapsto cw + d = u$ is also linear: $u = c(az+b) + d = acz + (bc + d)$

• all linear transformations are conformal on $\mathbb{C}$

• If $a = \rho e^{i\varphi}$ ($\rho \in \mathbb{R}_{>0}$), then $az + b = (\rho \cdot (e^{i\varphi} \cdot z)) + b$ - composition of above maps

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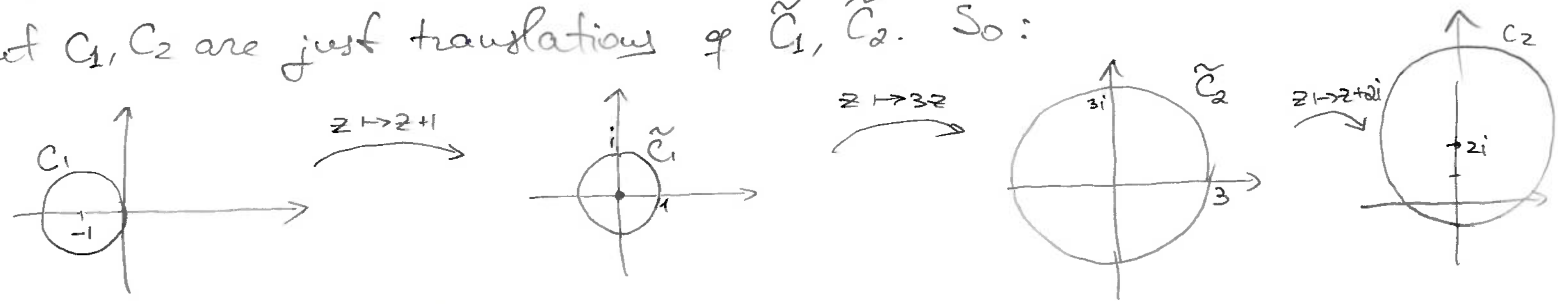
Ex2: a) Find a linear transformation mapping $C_1 = \{z : |z+1| = 1\}$ onto $C_2 = \{w : |w - 2i| = 3\}$

b) Find a linear transformation in a) mapping $0 \in C_1$ to $-3+2i \in C_2$.

a) If $\tilde{C}_1$ = unit circle $\{|z|=1\}$, $\tilde{C}_2$ = circle $\{|z|=3\}$, then clearly $\tilde{C}_2 = 3 \cdot \tilde{C}_1$.

But C_1, C_2 are just translations of $\tilde{C}_1, \tilde{C}_2$. So:



Hence, the map $z \mapsto 3(z+1)+2i = 3z+3+2i$ maps C_1 to C_2

(! it's highly non-unique).

b) Consider $f(z) = 3z+3+2i$ from above. Note that $f(0) = 3+2i$ is not as needed. But if we apply at the end rotation by π around the center $w_0 = 2i$ then it maps $3+2i \longmapsto -3+2i$.

[**Note**: Rotation by π around origin is $z \longmapsto -z$, hence, rotation by π around $w_0 \in \mathbb{C}$ is $w \mapsto w_0 - (w - w_0) = 2w_0 - w$.

So: $g(z) = 2 \cdot 2i - (3z+3+2i) = 2i - 3 - 3z$ is an example of such map $\blacksquare$

(again, the answer is non-unique!)

Let's now look at yet another basic transformation:

- inversion: $z \longmapsto f(z) = \frac{1}{z}$

  Note that $\frac{1}{z}$ is analytic on $\mathbb{C} \setminus \{0\}$ and is 1-to-1 on it.

  In polar coordinates $(\rho e^{i\varphi})^{-1} = \rho^{-1} \cdot e^{i(-\varphi)}$

  Finally, evoking extended cpx plane $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ we see $f(0) = \infty$, $f(\infty) = 0$,

  i.e. f is 1-to-1 on $\hat{\mathbb{C}}$.

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Under the inversion map:

- circles $C_r(0)$ are mapped to circles $C_{1/r}(0)$ BUT oppositely oriented
- inside & outside of $C_r(0)$ are mapped to outside & inside of $C_{1/r}(0)$
- lines passing through the origin are mapped to lines through 0.

Ex 3 (= [Lecture 5, Exercise 1]): Describe image of $C_1(1)=\{z: |z-1|=1\}$ under the inversion map

▸ $C_1(1)$ can be parametrized via $z = 1+e^{i\varphi} = (1+\cos\varphi)+i\sin\varphi, \quad 0\le\varphi<2\pi$

Then $w=\frac{1}{z} = \frac{1+\cos\varphi - i\sin\varphi}{(1+\cos\varphi)^2+\sin^2\varphi} = \frac{1+\cos\varphi - i\cdot\sin\varphi}{2(1+\cos\varphi)} = \frac{1}{2} - \frac{\sin\varphi}{2(1+\cos\varphi)}i$

So the image of $C_1(1)$ is on the vertical line $\{\mathrm{Re}\, w=\frac{1}{2}\}$ and can be shown to be all of it. So the answer is $\mathrm{Re}(w)=1/2$. ☐

The above exercise generalizes to:

Claim: Image of any line or circle on $\mathbb{C}$ under the inversion map is again either a line or circle.

▸ Recall the stereographic projection b/w the Riemann sphere $S^2=\{(x_1,x_2,x_3): x_1^2+x_2^2+x_3^2=1\}$ and the extended complex plane $\hat{\mathbb{C}}$. By [Lecture 5, Fact] the stereographic projection of any line or circle on $\mathbb{C}$ is a circle on S^2 (and the converse also holds).

[**Subclaim**: inversion map on $\mathbb{C}$ corresponds to the rotation of S^2 by angle π around x_1-axis

Given this result, clearly rotation of a circle on S^2 is another circle on S^2, hence, it is a stereographic proj. of line or circle.

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(Continuation)

It remains to prove the SubClaim.

But $(x_1, x_2, x_3) \in S^2$ corresponds to $z = \frac{x_1 + x_2 i}{1 - x_3}$, while its rotation as above is $(x_1, -x_2, -x_3) \in S^2$ which corresponds to $w = \frac{x_1 - x_2 i}{1 + x_3}$. It remains to show $w = \frac{1}{z}$, i.e. $\frac{x_1 - x_2 i}{1 + x_3} = \frac{1 - x_3}{x_1 + x_2 i}$, which follows from $x_1^2 + x_2^2 + x_3^2 = 1$

(see Example 4 on pp. 55-56 of textbook for an opposite argument)

Note: if one thinks of a line on $\mathbb{C}$ as a "circle of ∞ radius", then above claim says inversion map applied to any "generalized circle" is a "generalized circle"

! As circles are bounded and lines are not, we note that in the above claim, image of a "generalized circle" C under the inversion map is a line iff $0 \in C$.

Composing any of the above transformations gives rise to:

Def: A Möbius transformation (a.k.a. Fractional Linear Transform.) F.L.T. is a function $z \mapsto w = f(z) = \frac{az + b}{cz + d}$ with $a, b, c, d \in \mathbb{C}$ and $ad - bc \neq 0$

Note:
- if $ad = bc$, then f is just a constant
- $f'(z) = \frac{a(cz+d) - c(az+b)}{(cz+d)^2} = \frac{ad - bc}{(cz+d)^2} \neq 0 \Rightarrow f$ - conformal everywhere except pole
- if $c = 0$, then $f(z) = \frac{a}{d} z + \frac{b}{d}$ - linear transformation
- if $c \neq 0$, then $\frac{az+b}{cz+d} = \frac{\frac{a}{c}(cz+d) + b - \frac{ad}{c}}{cz+d} = \frac{a}{c} + \left(b - \frac{ad}{c}\right) \cdot \frac{1}{cz+d}$

  which is always a composition of above four basic transformations.

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As any Möbius transformation is a composition of above four elementary types (translation, magnification, rotation, inversion), we get:

Theorem: Let f be a Möbius transformation. Then:
- f maps extended $\hat{\mathbb{C}}$ one-to-one to itself
- f maps "generalized circles" to "generalized circles"
- f is conformal at each point except its pole

open disk of radius 1 centered at 1.

Ex 4: Find the image of $D_1(1) = \{z \in \mathbb{C} : |z-1| \le 1\}$ under $z \mapsto w = f(z) = \frac{3z}{2z-4}$

▸ Know by theorem that $f(C_1(1))$ is either a circle or line. But $\infty = f(2)$ and $2 \in C_1(1) \Rightarrow$ it is a line. As any line is determined by two points on it, it suffices to compute image of two points on $C_1(1)$:

$f(0) = 0$

$f(1+i) = \frac{3(1+i)}{-2+2i} = \frac{(3+3i)(-2-2i)}{4+4} = \frac{-12i}{8} = -\frac{3}{2}i$

So: $C_1(1)$ is mapped to the imaginary axis $\{w : \text{Re}\, w = 0\}$.

Therefore, interior of $C_1(1)$ is mapped either to $\{w : \text{Re}\, w > 0\}$ or $\{w : \text{Re}\, w < 0\}$. To decide which one, need to compute image of some point inside $C_1(1)$. E.g. $f(1) = -\frac{3}{2}$. So: $\{w \in \mathbb{C} \mid \text{Re}\, w < 0\}$ ∎

Ex 5: Find a conformal map of the unit disk $D_1(0) = \{z \in \mathbb{C} : |z| < 1\}$ onto the upper half plane $\mathcal{H} = \{w \in \mathbb{C} : \text{Im}\, w > 0\}$.

Idea: Look for F.L.T mapping $C_1(0)$ to $\mathbb{R} \subseteq \mathbb{C}$ and then make sure interior get mapped to $\mathcal{H}$ and not $-\mathcal{H}$.

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(Solution of Ex 5)

As $f(C_1(0))$ - line containing 0 $\Rightarrow$ there are two points $z_1, z_3 \in C_1(0)$ such that $f(z_1)=0$, $f(z_3)=\infty$. We also look for $f(z)=\frac{az+b}{cz+d}$.

For simplicity, take $z_1=-1$, $z_3=1$. Then $a-b=0$, $c+d=0$, and we see that $f(z)=A\cdot\frac{z+1}{z-1}$ for some nonzero $A\in\mathbb{C}$ to guarantee that $f(C_1(0))$ is a line passing through the origin.

To check it's the horizontal line compute $f(z_2)$ for any other $z_2\in C_1(0)$

E.g. $f(i)=A\cdot\frac{i+1}{i-1}=\frac{A}{i} \Rightarrow \frac{A}{i}=t\in\mathbb{R} \Rightarrow A=it$, t - real.

Finally, to guarantee interior is mapped to $\mathcal{H}$ not $-\mathcal{H}$, compute $f(z_4)$ for any z_4 inside $C_1(0)$. E.g. $f(0)=A\cdot(-1)=-it\in\mathcal{H} \Leftrightarrow t<0$.

So: One of possible choices is $\boxed{f(z)=-i\cdot\frac{z+1}{z-1}=i\cdot\frac{1+z}{1-z}}$