# Lecture #38 (§7.4)

**Ex1** : a) Is composition of Möbius transformations again Möbius?

b) Is inverse of any Möbius transformation again Möbius?

[ Rmk : These are called "group properties"

a) Let $L_1(z) = \frac{az+b}{cz+d}$ , $L_2(z) = \frac{\tilde{a}z+\tilde{b}}{\tilde{c}z+\tilde{d}}$ with $ad-bc \neq 0$ , $\tilde{a}\tilde{d}-\tilde{b}\tilde{c} \neq 0$. Then:

$$(L_1 \circ L_2)(z) = \frac{a \cdot \frac{\tilde{a}z+\tilde{b}}{\tilde{c}z+\tilde{d}} + b}{c \cdot \frac{\tilde{a}z+\tilde{b}}{\tilde{c}z+\tilde{d}} + d} = \frac{(a\tilde{a}+b\tilde{c})z + (a\tilde{b}+b\tilde{d})}{(c\tilde{a}+d\tilde{c})z + (c\tilde{b}+d\tilde{d})}$$

Also: $(a\tilde{a}+b\tilde{c})(c\tilde{b}+d\tilde{d}) - (c\tilde{a}+d\tilde{c})(a\tilde{b}+b\tilde{d}) =$

$$\cancel{a\tilde{a}c\tilde{b}} + a\tilde{a}d\tilde{d} + b\tilde{b}c\tilde{c} + \cancel{bd\tilde{c}\tilde{d}} - \cancel{ac\tilde{a}\tilde{b}} - bc\tilde{a}\tilde{d} - ad\tilde{b}\tilde{c} - \cancel{bd\tilde{c}\tilde{d}} =$$

$$(ad-bc)(\tilde{a}\tilde{d}-\tilde{b}\tilde{c}) \neq 0$$

Hence, the composition of Möbius is again Möbius!

b) If $w = L_1(z) = \frac{az+b}{cz+d}$ , then $az+b = cwz + dw \Rightarrow z = \frac{dw-b}{-cw+a}$

Thus, the inverse map is $L_1^{-1}(w) = \frac{d \cdot w + (-b)}{(-c) \cdot w + a}$ with $ad-bc \neq 0$

which is again Möbius!

[ Rmk : Last time we verified that every Möbius transformation is a composition of 4 basic maps: rotation, magnification, parallel translation, and inverse. But by Ex1a) we see that the converse is also true, i.e. any composition of above is an F.L.T.

[ Note : * $\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \frac{1}{ad-bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$ which is compatible with Ex1b) above

* $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot \begin{pmatrix} \tilde{a} & \tilde{b} \\ \tilde{c} & \tilde{d} \end{pmatrix} = \begin{pmatrix} a\tilde{a}+b\tilde{c} & a\tilde{b}+b\tilde{d} \\ c\tilde{a}+d\tilde{c} & c\tilde{b}+d\tilde{d} \end{pmatrix}$ which is compatible with Ex1a)

* Here we note that $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ & $\lambda \cdot \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} \lambda a & \lambda b \\ \lambda c & \lambda d \end{pmatrix}$ determine same $L$

In fact, the group of Möbius transformations is $PGL_2(\mathbb{C})$

# Lecture #38

Last time we found a conformal map of the unit disk $D_1(0) = \{|z|<1\}$ onto the upper half plane $\mathcal{H} = \{\operatorname{Im} w > 0\}$: $\boxed{f(z) = i \cdot \frac{1+z}{1-z}}$

Let us now utilize this to solve the following Dirichlet problem:

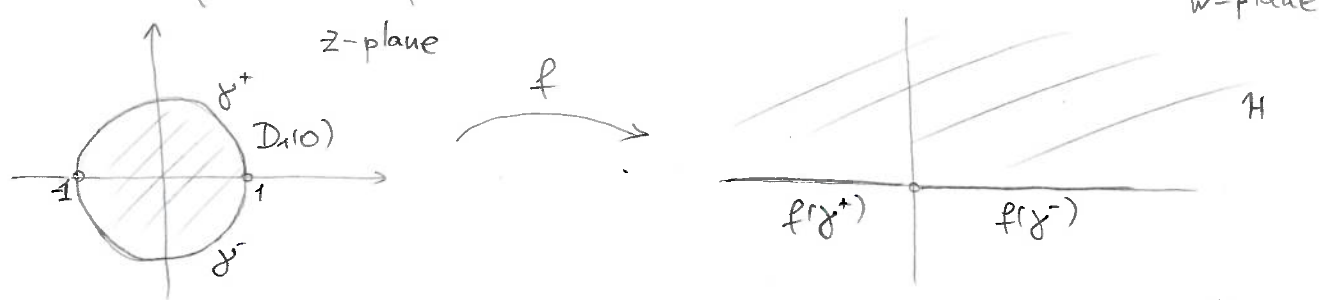
Ex2 (Ex1 of Sect 7.1): Find a harmonic function $\phi(x,y)$ on $D_1(0)$ such that

$$\phi(x,y) \begin{cases} \to +1 & \text{in } \mathcal{H} \cap C_1(0) = \text{upper half-circle} = \gamma^+ \\ \to -1 & \text{in } (\mathbb{C}\setminus\mathcal{H}) \cap C_1(0) = \text{lower half-circle} = \gamma^- \end{cases}$$

▸ Consider above map $z \mapsto w = f(z) = i \cdot \frac{1+z}{1-z}$ mapping $D_1(0)$ onto $\mathcal{H}$.

As $f(-1)=0,\ f(i)=-1,\ f(1)=\infty$, we get $f(\gamma^+) = \mathbb{R}_{<0}$

$f(-1)=0,\ f(-i)=1,\ f(1)=\infty \Rightarrow \quad f(\gamma^-) = \mathbb{R}_{>0}$




Pick a harmonic $\Phi$ on $\mathcal{H}$ with boundary condition $\Phi \begin{cases} \to -1 & \text{on } \mathbb{R}_{>0} \\ \to +1 & \text{on } \mathbb{R}_{<0} \end{cases}$

by solving a wedge problem (see Lecture 11). For example, we look for $\Phi = A \cdot \arg_{-\pi/2} w + B$ so that $B=-1$ & $A\cdot\pi + B = 1 \Rightarrow A = 2/\pi,\ B=-1$

Thus, can take $\boxed{\Phi(w) = \frac{2}{\pi}\arg_{-\pi/2}(w) - 1}$

Then, by [Lecture 36, Theorem 2], the composition $\phi := \Phi \circ f$ solves original problem

Here: $\phi(x,y) = \frac{2}{\pi}\arg_{-\pi/2}\left(i \cdot \frac{1+z}{1-z}\right) - 1 = \frac{2}{\pi}\left(\arg_{-\pi/2}\left(\frac{1+z}{1-z}\right) + \frac{\pi}{2}\right) - 1 = \boxed{\frac{2}{\pi}\arg_{-\pi/2}\left(\frac{1+z}{1-z}\right)}$

Evoking $z = x+iy$, we have $\frac{1+z}{1-z} = \frac{1+x+iy}{1-x-iy} = \frac{(1+x+iy)(1-x+iy)}{(1-x)^2+y^2} = \frac{1-x^2-y^2+i\cdot 2y}{(1-x)^2+y^2}$

$\Rightarrow \boxed{\phi(x,y) = \frac{2}{\pi}\tan^{-1}\left(\frac{2y}{1-x^2-y^2}\right)}$

∎

[Note: This solution is much easier than Poisson integral formula (Lecture 18)

# Lecture # 38

As we saw above, the key was to find a Möbius map $f : C_1(0) \xrightarrow[1:1]{\text{onto}} \mathbb{R}$

**Question 1**: Find a Möbius map which maps any given "generalized circle" $C_1$ onto another given "generalized circle" $C_2$.

From previous class we know that Möbius maps always preserve class of generalized circles (i.e. circles & lines). As every generalized circle is determined by 3 points on it (for lines – need just 2, i.e. take $3^{rd}$ one $\infty$), the above question can be upgraded to:

**Question 2**: Given any three distinct points $z_1, z_2, z_3 \in \hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ in $z$-plane and three distinct points $w_1, w_2, w_3 \in \hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ in $w$-plane, find Möbius $f$ such that $f(z_1) = w_1$, $f(z_2) = w_2$, $f(z_3) = w_3$.

To answer this, let's first find a Möbius map $L_1$ so that

$$\boxed{L_1 : z_1 \mapsto 0, \ z_2 \mapsto 1, \ z_3 \mapsto \infty}$$

Arguing as last time, we see that $L_1(z_1) = 0$, $L_1(z_3) = \infty$ determine $L_1$ is of the form $L_1(z) = A \cdot \frac{z - z_1}{z - z_3}$ for some $A \in \mathbb{C} \setminus \{0\}$, while $L(z_2) = 1$ then determines $A \cdot \frac{z_2 - z_1}{z_2 - z_3} = 1 \Rightarrow A = \frac{z_2 - z_3}{z_2 - z_1}$ and so unique such $L_1$ is:

$$\boxed{L_1(z) = \frac{z - z_1}{z - z_3} \cdot \frac{z_2 - z_3}{z_2 - z_1}}$$ ← This answers Question 2 when $w_1 = 0$, $w_2 = 1$, $w_3 = \infty$

But we can also consider another Möbius map $L_2$, such that

$$\boxed{L_2 : w_1 \mapsto 0, \ w_2 \mapsto 1, \ w_3 \mapsto \infty}$$

It's given by the formula just as above:

$$\boxed{L_2(w) = \frac{w - w_1}{w - w_3} \cdot \frac{w_2 - w_3}{w_2 - w_1}}$$

Then, the composition $f = L = \boxed{L_2^{-1} \circ L_1}$ is a Möbius map (by Ex 1) which satisfies

$$\boxed{f(z_1) = w_1, \quad f(z_2) = w_2, \quad f(z_3) = w_3}$$

answering Question 2 above

# Lecture #38

In fact, the above solution $f$ to Question 2 is unique:

**Lemma 1**: If $S, T$ are Möbius maps such that $S(z)=T(z)$ at 3 distinct points $z_1, z_2, z_3$, then $S=T$!

▶ Consider $L := S^{-1} \circ T$ - also Möbius by Ex1. Then $L(z)=z$ at 3 distinct points. The next result implies that $L=$ identity (and so $S=T$).

**Claim** (Exercise 6, p. 392): Any Möbius map that fixes 3 distinct points is Id.

▷ Let $L(z) = \frac{az+b}{cz+d}$.

* If $c=0 \Rightarrow \frac{az+b}{d} = z$ is equivalent to $(a-d)z+b=0 \Rightarrow$ has $>1$ solution only iff $a=d$, $b=0 \Rightarrow L(z) = \frac{az}{z} = z$ i.e. $L = Id$

* If $c \neq 0 \Rightarrow \frac{az+b}{cz+d} = z$ is equivalent to $c \cdot z^2 + (d-a)z - b = 0$ which has $\leq 2$ solutions $\Rightarrow$ contradiction

□

**Ex3**: Let $L$ be a Möbius map such that $L(0)=4$, $L(1)=3$, $L(-1)=\infty$. Find $L(z)$

▶ We apply construction of previous page with $\begin{cases} z_1=0, \ z_2=1, \ z_3=-1 \\ w_1=4, \ w_2=3, \ w_3=\infty \end{cases}$ to get:

$$L_1(z) = \frac{z-z_1}{z-z_3} \cdot \frac{z_2-z_3}{z_2-z_1} = \frac{z}{z+1} \cdot 2$$

$$L_2(w) = \frac{w-w_1}{w-w_3} \cdot \frac{w_2-w_3}{w_2-w_1} = \frac{w-4}{w-\infty} \cdot \frac{3-\infty}{-1} = 4-w$$

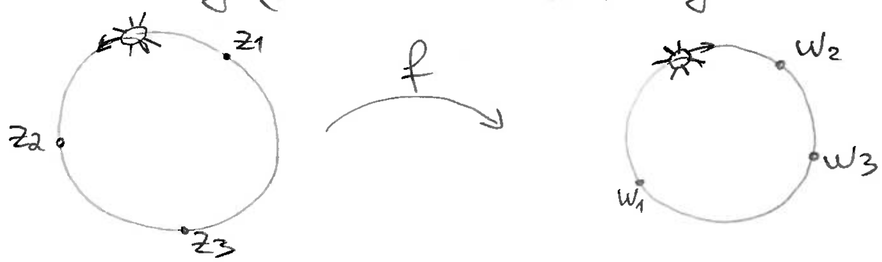
As $L_1(z)$ & $L_2(w)$ must equal with $w = L(z)$ - looked after map, we conclude $4 - w = \frac{2z}{z+1} \Rightarrow w = 4 - \frac{2z}{z+1} = \boxed{\frac{2z+4}{z+1}}$

**Ex 4** (Exercise 1, p. 403): Let $f_1(z) = \frac{z+2}{z+3}$, $f_2(z) = \frac{z}{z+1}$. Find $f_1^{-1}(f_2(z))$.

▶ $f_1^{-1}(z) = \frac{3z-2}{-z+1}$ by Ex 1b) $\Rightarrow f_1^{-1}(f_2(z)) = \frac{3 \cdot \frac{z}{z+1} - 2}{-\frac{z}{z+1} + 1} = \frac{z-2}{1} = \boxed{z-2}$

Lecture #38

Rmk ("bug test"): Now that we know how to construct a Möbius map $f$ that maps a generalized circle $C_1$ to a generalized circle $C_2$ by mapping 3 points $z_1, z_2, z_3 \in C_1$ to specified $w_1, w_2, w_3 \in C_2$, it still remains to decide if $f$ maps interior of $C_1$ to interior or exterior of $C_2$ (note: if $C_1$ or $C_2$ is a line, then we have just two half planes). To this end, imagine you have a "bug" crawling from $z_1$ to $z_2$ and then look at its "image" as crawling from $w_1$ to $w_2$, e.g.




corresponds to $f(\text{interior}) = \text{exterior}$

Let us conclude today's class with two important results (see proofs in the supplemental note):

**Theorem 1**: The group of automorphisms of the unit disk $D_1(0)$, i.e.

$$\mathrm{Aut}(D_1(0)) = \{\text{all analytic onto 1-to-1 maps } D_1(0) \to D_1(0)\}$$

consists of the following Möbius transformations:

$$\left\{ \lambda \cdot \frac{z-a}{1-\bar{a}z} \;\middle|\; \underbrace{a \in D_1(0)}_{\text{i.e. } |a|<1} \text{ and } |\lambda| = 1 \right\}$$

[Note: We shall encounter such Möbius maps later this week

The proof relies on another important result:

**Theorem 2** ("Schwarz lemma"): Let $f: D_1(0) \to D_1(0)$ be an analytic map such that $f(0)=0$. Then:

A) $|f(z)| \le |z|$ for any $z \in D_1(0)$

B) $|f'(0)| \le 1$.

Moreover, if equality holds in A) for some $z \neq 0$, or holds in B), then in fact $f(z) = \lambda \cdot z$ with $|\lambda| = 1$, i.e. $f$ - rotation of $D_1(0)$