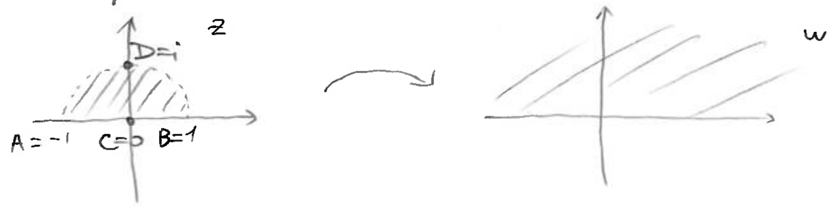
Lecture #39

Ex 1: Find a conformal map that maps $D = \{z : |z|<1 \ \& \ \operatorname{Im} z > 0\}$ onto $H = \{w : \operatorname{Im} w > 0\}$

Want:

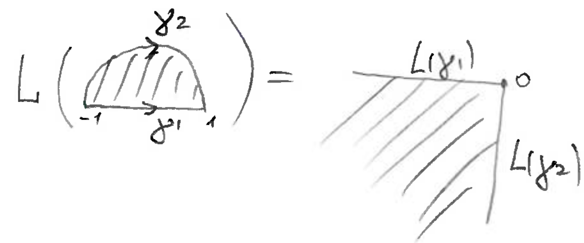


Step 1: Use a Möbius map L that will map D onto some wedge

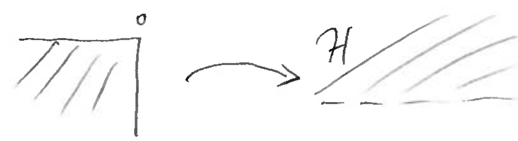
E.g. if $A \mapsto 0$, $B \mapsto \infty$, then $L = \text{const} \cdot \frac{z+1}{z-1}$. Consider such map with const=1:

$$L(z) = \frac{z+1}{z-1}$$

Note: $L(0) = -1 \Rightarrow L(\text{segment } AB) = \mathbb{R}_{\leq 0}$
$L(i) = -i \Rightarrow L(\text{arc } ADB) = i \cdot \mathbb{R}_{\leq 0}$
$\Rightarrow$

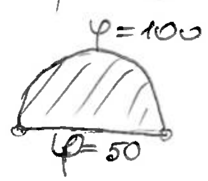


Step 2: Apply $z \mapsto z^2 = (-z)^2$ to map above wedge

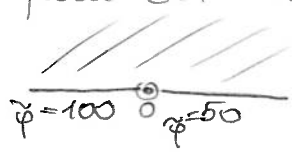


So: One of possible answers is $f(z) = \left(\frac{z+1}{z-1}\right)^2$ ∎

Ex 2: Solve Dirichlet problem on $D_1(0) \cap H$ with boundary conditions



Apply $f(z) = \left(\frac{z+1}{z-1}\right)^2$ from Ex 1 to map above onto H. Then boundary conditions read



Look for harmonic $\tilde{\varphi}(w) = A \cdot \arg_{-\pi/2} w + B$. Then $B = 50$, $A \cdot \pi + 50 = 100 \Rightarrow A = \frac{50}{\pi}$.

So: $\varphi(z) = \frac{50}{\pi} \arg_{-\pi/2}\left(\left(\frac{z+1}{z-1}\right)^2\right) + 50$ is an example. ∎

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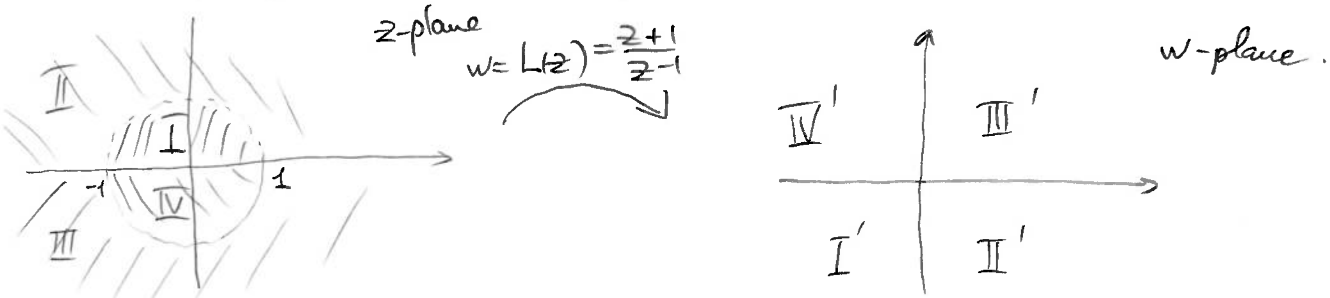
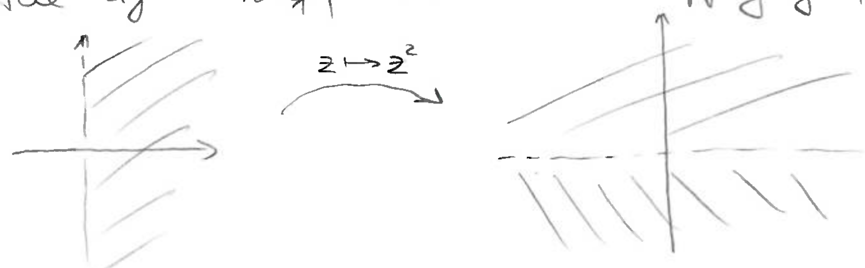
Back to Step 1 from solution of Ex1 we note that

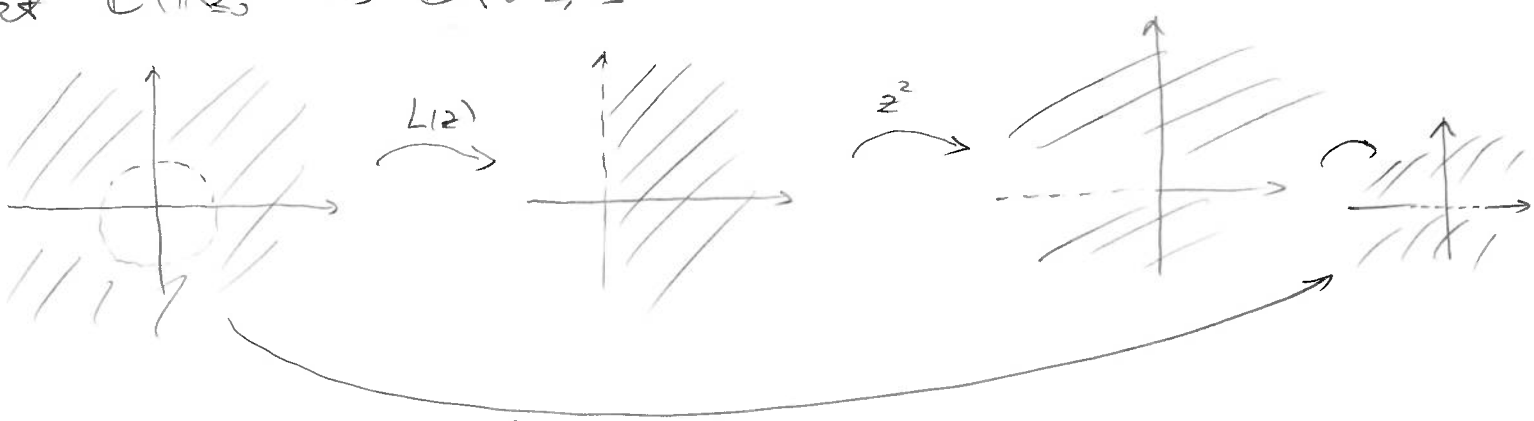


In particular, the exterior of unit circle, i.e. $II \cup III$ are mapped to the right half-plane $\mathrm{Re}\, w > 0$. Applying further $z \mapsto z^2$, we get



$= \mathbb{C} \setminus \mathbb{R}_{\leq 0}$ – all cpx plane without $\mathbb{R}_{\leq 0}$.

Applying further the Möbius map $z \mapsto \frac{z+1}{-z+1}$ s.t. $0 \mapsto 1$, $\infty \mapsto -1$, we see that $\mathbb{C} \setminus \mathbb{R}_{\leq 0} \longrightarrow \mathbb{C} \setminus [-1, 1]$



composition: $z \mapsto -\frac{1}{2}\left(z + \frac{1}{z}\right)$

|| <u>Def</u>: $z \mapsto J(z) = \frac{1}{2}\left(z + \frac{1}{z}\right)$ is a (famous) <u>Jukovsky map</u>.

In polar coordinates, if $z = re^{i\theta}$ then $J(z = re^{i\theta}) = \frac{r + r^{-1}}{2} \cos\theta + i \cdot \frac{r - r^{-1}}{2} \sin\theta$

<u>Note</u>: Image of any circle $|z| = r$ is now an ellipse $\left(\frac{x}{\frac{r+r^{-1}}{2}}\right)^2 + \left(\frac{y}{\frac{r-r^{-1}}{2}}\right)^2 = 1$ except for unit circle $|z| = 1$ which is mapped to the segment $[-1, 1]$

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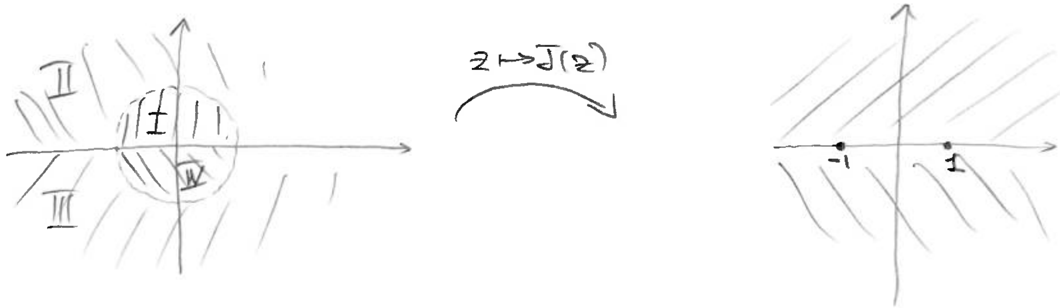


$\mathcal{H} = J(\text{II}) = J(\text{IV})$

$-\mathcal{H} = J(\text{I}) = J(\text{III})$

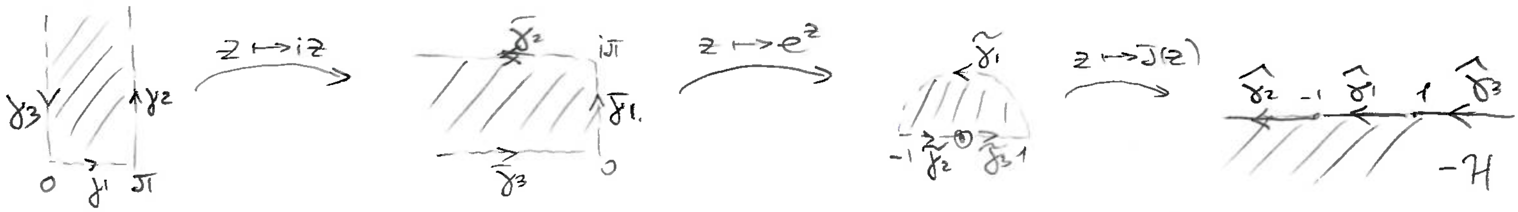
Note: $[1,+\infty)$ is the image of $(0,1]$.
$(-\infty,-1]$ is the image of $[-1,0)$
$[-1,1]$ is the image of both top and bottom halves of unit circle

Also: $J(z) = J(1/z)$ and the map $z \mapsto \frac{1}{z}$ maps $\text{II} \leftrightarrow \text{IV}$, $\text{I} \leftrightarrow \text{III}$.

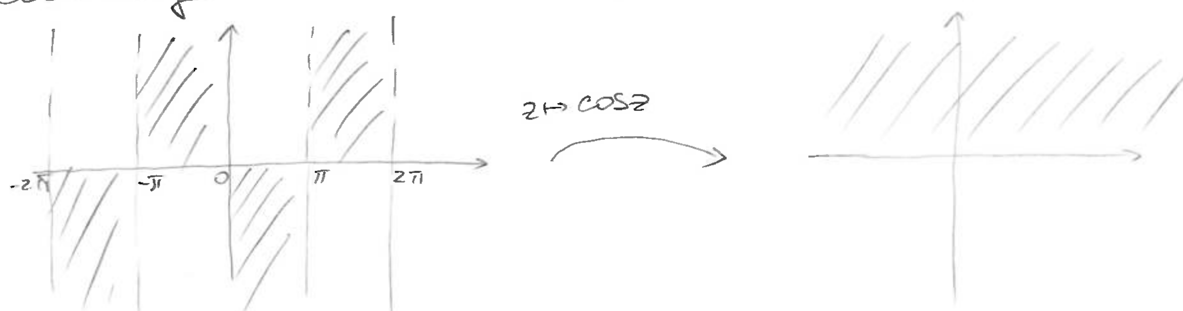
Recall that $\cos z = \dfrac{e^{iz}+e^{-iz}}{2}$, so that

$$\boxed{\cos(z) = J(e^{iz})}$$

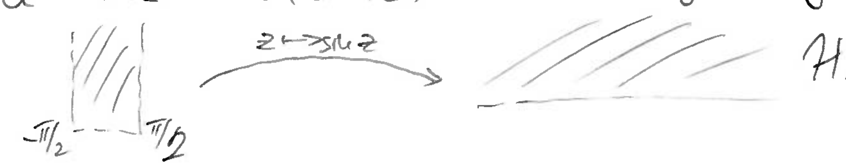
Let's now visualize the geometry of the cosine map. We start with a half of vertical strip $\{z \in \mathbb{C} : \operatorname{Im} z > 0,\ 0 \le \operatorname{Re} z < \pi\}$



Combining this with $\cos(-z) = \cos z$ and $\cos(z+\pi) = -\cos z$, we get



Combining this with $\sin z = -\cos(z + \pi/2)$ one also gets geometry of sin-map
In particular



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Let's just conclude with some useful toolbox of conformal maps