Lecture #40

Today: 1) Finish §7.4 of the book

2) Discuss "Summation of Series" back from pp. 327–328.

Def: Two points z_1, z_2 are said to be symmetric w.r.t. a generalized circle C if and only if any generalized circle passing through z_1 & z_2 intersects C orthogonally.

Rmks: a) If C is a line, then this is equivalent to obvious symmetry clearly.

b) If C - circle, z_1 - its center, then $(z_1, z_2=\infty)$ are clearly symmetric.

Theorem 1 (Symmetry principle): Let $w=f(z)$ be any Möbius transformation. Then two points z_1, z_2 are symmetric w.r.t. generalized circle C if and only if their images $w_1=f(z_1)$, $w_2=f(z_2)$ are symmetric w.r.t. $f(C)$.

▷ Obvious as any Möbius map is conformal and preserves the class of generalized circles ■

Question: Given a circle $C_R(a)=\{z: |z-a|=R\}$ and $\alpha\in\mathbb{C}$ find α^* so that the pair (α,α^*) is symmetric w.r.t. $C_R(a)$. Any explicit formula?

We shall utilize theorem above to find a Möbius map L that maps $C_R(a)$ to a line, e.g. $\mathbb{R}\subseteq\mathbb{C}$, for which notion of symmetry is easy. To this end, pick 3 points on $C_R(a)$ that are mapped to $0, 1, \infty$, e.g.

$$\boxed{L: a-R\mapsto 0,\quad a+R\mapsto\infty,\quad a+Ri\mapsto 1}$$

which is uniquely constructed as in Lecture 38:

$$L(z)=\frac{z-(a-R)}{z-(a+R)}\cdot\frac{(a+Ri)-(a+R)}{(a+Ri)-(a-R)}=\boxed{i\cdot\frac{z-(a-R)}{z-(a+R)}}$$

Then: $L(\alpha), L(\alpha^*)$ must be symmetric w.r.t. $\mathbb{R} \iff \boxed{T(\alpha^*)=\overline{T(\alpha)}}$

reflecting $x+iy$ w.r.t. $\mathbb{R}$ gives $x-iy=\overline{x+iy}$

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Let's solve the above equation $T(\alpha^*) = \overline{T(\alpha)}$ to express α^* via α, a, R.

$$T(\alpha^*) = i \cdot \frac{\alpha^* - a + R}{\alpha^* - a - R}, \quad T(\alpha) = i \cdot \frac{\alpha - a + R}{\alpha - a - R}$$

so that

$$i \cdot \frac{\alpha^* - a + R}{\alpha^* - a - R} = \overline{i \cdot \frac{\alpha - a + R}{\alpha - a - R}} = -i \cdot \frac{\bar\alpha - \bar a + R}{\bar\alpha - \bar a - R}$$

$$\Updownarrow$$

$$\frac{\alpha^* - a + R}{\alpha^* - a - R} = - \frac{\bar\alpha - \bar a + R}{\bar\alpha - \bar a - R}$$

$$\Updownarrow$$

$$(\bar\alpha - \bar a - R)(\alpha^* - a + R) = -(\bar\alpha - \bar a + R)(\alpha^* - a - R)$$

$$\Updownarrow$$

$$(\bar\alpha - \bar a - R)\alpha^* - \bar\alpha a + a\bar a + \cancel{aR} + \cancel{\bar\alpha R} - \bar a R - R^2 = -(\bar\alpha - \bar a + R)\alpha^* + \bar\alpha a - a\bar a + \cancel{aR} + \cancel{\bar\alpha R} - \bar a R + R^2$$

$$\Updownarrow$$

$$2(\bar\alpha - \bar a)\alpha^* = 2\bar\alpha a - 2a\bar a + 2R^2$$

$$\Updownarrow$$

$$\boxed{\alpha^* = \frac{R^2}{\bar\alpha - \bar a} + a}$$

<u>Upshot</u>: The unique point on $\hat{\mathbb{C}}$ which is symmetric to α w.r.t. $C_R(a)$ is
$$\alpha^* = \frac{R^2}{\bar\alpha - \bar a} + a$$

Geometric meaning of this point:

1) $|\alpha^* - a| = R^2/|\bar\alpha - \bar a| = R^2/|\alpha - a|$, i.e. product of distances from the center a to α, α^* equals R^2.

2) $\arg(\alpha^* - a) = \arg\left(\frac{1}{\overline{\alpha - a}}\right) = \arg(\alpha - a)$.

<u>So</u>: Points α^* and α lie on the same ray from the center a and the product of distances to center is R^2.

<u>Ex 1</u>: Does there exist a Möbius map taking imaginary axis $i\mathbb{R}$ onto the unit circle $|w| = 1$ and mapping $1 \mapsto 2$, $-1 \mapsto \frac{1}{3}$?

<u>No</u>: points $1, -1$ are symmetric w.r.t. $i\mathbb{R}$, while $2, \frac{1}{3}$ are not symmetric w.r.t. unit circle by above.

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Ex 2 : Find all Möbius transformations that map unit disc $|z|\leq 1$ onto $|w|\leq 1$

Any such map $z\mapsto w=f(z)$ must map $|z|=1$ onto $|w|=1$ and preserve interior. In particular, there is some α inside s.t. $f(\alpha)=0$. But then

$$\left.\begin{array}{l} f(\alpha^*)=\infty \text{ by Theorem 1} \\ \text{Note : } \alpha^* = 1/\bar{\alpha} \end{array}\right\} \Rightarrow f(z) = \underset{\substack{\uparrow \\ \text{some nonzero constant}}}{c}\cdot\frac{z-\alpha}{z-1/\bar{\alpha}} = (-c\bar{\alpha})\cdot\frac{z-\alpha}{1-\bar{\alpha}\cdot z}$$

As $1=|f(1)| = |-c\bar{\alpha}|\cdot\left|\frac{1-\alpha}{1-\bar{\alpha}}\right| \Rightarrow |-c\bar{\alpha}|=1 \Rightarrow$ constant $\lambda := -c\bar{\alpha}$ has $|\lambda|=1$.

Thus : $\boxed{f(z)=\lambda\cdot\frac{z-\alpha}{1-\bar{\alpha}z} \text{ with } \alpha\in D_1(0),\ |\lambda|=1}$

Moreover, any such map satisfies conditions as it maps unit circle on itself:

$$f(e^{i\theta}) = \lambda\cdot\frac{e^{i\theta}-\alpha}{1-\bar{\alpha}e^{i\theta}} = \underbrace{\lambda e^{-i\theta}}_{|\lambda e^{-i\theta}|=1}\cdot\underbrace{\frac{e^{i\theta}-\alpha}{e^{-i\theta}-\bar{\alpha}}}_{|\ldots|=1 \text{ as denominator is cpx conjugate to numerator}}$$

and $f(\alpha)=0$ hence interior $\mapsto$ interior ∎

Rmk : In Lecture #38 - supplemental we actually proved that any analytic map $D_1(0) \overset{\text{1-to-1}}{\underset{\text{onto}}{\circlearrowleft}}$ is of above form! This should be viewed as a special case of the Riemann Mapping Theorem.

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We conclude today's lecture with another important application of residue theory – for summation of series. This is discussed to some extent at pp. 327-328 of your book (exercises to §6.3).

Ex 3: Evaluate $\sum\limits_{k=-\infty}^{+\infty} \frac{1}{k^2+1}$

To approach this problem, we shall use the following result:

Theorem 2: Let $f(z)=\frac{P(z)}{Q(z)}$ be a rational function with $\deg Q \geq \deg P+2$, and without poles at $\mathbb{Z}=\{\text{all integers}\}$. Then:

$$\lim_{N\to+\infty} \sum_{k=-N}^{N} f(k) = -\sum_{z_0-\text{pole of } f(z)} \text{Res}_{z_0}\left(\pi\cdot f(z)\cdot \cot(\pi z)\right)$$

Solution of Ex2

Take $f(z)=\frac{1}{z^2+1}$ which has poles at $\pm i$.

$$\text{Res}_i\left(\frac{\pi}{z^2+1}\cot(\pi z)\right)=\frac{\pi\cot(\pi z)}{z+i}\Big|_{z=i}=\frac{\pi}{2i}\cdot\frac{\cos(\pi i)}{\sin(\pi i)}=\frac{\pi}{2i}\cdot\frac{\frac{e^{\pi i^2}+e^{-\pi i^2}}{2}}{\frac{e^{\pi i^2}-e^{-\pi i^2}}{2i}}=\frac{\pi}{2}\cdot\frac{e^{-\pi}+e^{\pi}}{e^{-\pi}-e^{\pi}}$$

$$\text{Res}_{-i}\left(\frac{\pi}{z^2+1}\cot(\pi z)\right)=\frac{\pi\cot(\pi z)}{z-i}\Big|_{z=-i}\overset{\cot-\text{odd}}{=\!=}\frac{\pi}{2}\cdot\frac{e^{-\pi}+e^{\pi}}{e^{-\pi}-e^{\pi}}$$

So: $$\sum_{k=-\infty}^{+\infty}\frac{1}{k^2+1}=\lim_{N\to+\infty}\sum_{k=-N}^{N}\frac{1}{k^2+1}=\pi\cdot\frac{e^{\pi}+e^{-\pi}}{e^{\pi}-e^{-\pi}}=\boxed{\pi\coth(\pi)}$$

A slight variation of above result allows to treat alternating signs:

Theorem 3: Let $f(z)=\frac{P(z)}{Q(z)}$ be a rational function with $\deg Q\geq \deg P+2$, and without poles at $\mathbb{Z}$. Then:

$$\lim_{N\to+\infty}\sum_{k=-N}^{N}(-1)^k f(k) = -\sum_{z_0-\text{pole of } f(z)}\text{Res}_{z_0}\left(\pi f(z)\cdot\frac{1}{\sin(\pi z)}\right)$$

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The condition that $f(z)$ has no poles at $\mathbb{Z}$ can be easily resolved as illustrated in the following classical example:

Ex 4: a) Show that $\sum_{k=1}^{\infty} \frac{1}{k^2} = \frac{\pi^2}{6}$

b) Show that $\sum_{k=1}^{\infty} \frac{(-1)^k}{k^2} = -\frac{\pi^2}{12}$

Proof of Theorem 2

▶ Consider the function $g(z) = \pi \cdot f(z) \cdot \cot(\pi z)$. Its singularities are:

- poles of $f(z)$
- simple poles of $\cot(\pi z)$ which are $\mathbb{Z}$

Moreover, for any integer k, we have:

$$\mathrm{Res}_k\, g(z) = \mathrm{Res}_k \left(\frac{\pi \cos(\pi z) f(z)}{\sin(\pi z)} \right) = \frac{\pi \cos(\pi z) f(z)}{(\sin(\pi z))'} \Bigg|_{z=k} = f(k)$$

Pick a contour Γ_N which is a positively oriented square with vertices $(N+\frac{1}{2})(1+i)$, $(N+\frac{1}{2})(-1+i)$, $(N+\frac{1}{2})(-1-i)$, $(N+\frac{1}{2})(1-i)$.

$$\boxed{\text{Claim: There is some constant } M \in \mathbb{R}_{>0} \text{ s.t. } |\pi \cot(\pi z)| \leq M \quad \forall z \in \Gamma_N \ \forall N \in \mathbb{Z}.}$$

Using this Claim and $\deg Q \geq \deg P + 2$, we see that

$$\oint_{\Gamma_N} g(z)\,dz \longrightarrow 0 \quad \text{as } N \to +\infty$$

Applying Cauchy Residue theorem then implies the result:

$$\boxed{\lim_{N \to \infty} \sum_{k=-N}^{N} f(k) + \sum_{z_0 \text{-pole of } f(z)} \mathrm{Res}_{z_0}(g(z)) = 0}$$

Proof of Theorem 3 is completely analogous, but need to integrate $g(z) = \frac{\pi f(z)}{\sin(\pi z)}$ over the same contour Γ_N. ■

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Proof of Claim

Case 1: $z = x+iy$ with $y > \frac{1}{2}$

$$|\cot(\pi z)| = \left|\frac{e^{i\pi z}+e^{-i\pi z}}{e^{i\pi z}-e^{-i\pi z}}\right| \le \frac{|e^{i\pi z}|+|e^{-i\pi z}|}{|e^{-i\pi z}|-|e^{i\pi z}|} = \underbrace{\frac{1+e^{-2\pi y}}{1-e^{-2\pi y}}}_{\text{decreasing on } [\frac{1}{2},\infty)} \le \underbrace{\frac{1+e^{-\pi}}{1-e^{-\pi}}}$$

Case 2: $z = x+iy$ with $y < -\frac{1}{2}$

Same bound $|\cot(\pi z)| \le \underbrace{\frac{1+e^{-\pi}}{1-e^{-\pi}}}$ (as $\cot(\pi z)$ - odd function)

Case 3: $z = N+\frac{1}{2}+yi$ with $-\frac{1}{2} \le y \le \frac{1}{2}$

$$|\cot(\pi z)| = |\cot(\tfrac{\pi}{2}+i\pi y)| = |\tan(i\pi y)| = |\tanh(\pi y)| \quad \text{where } \tanh\theta = \frac{e^{2\theta}-1}{e^{2\theta}+1}.$$

As $\tanh(\theta)$ is increasing function (check $\frac{d}{d\theta}\tanh\theta > 0$), we get:

$$|\cot(\pi z)| \le \tanh(\pi/2) = \underbrace{\frac{e^{\pi}-1}{e^{\pi}+1}}$$

Case 4: $z = -N-\frac{1}{2}+yi$, $-\frac{1}{2} \le y \le \frac{1}{2}$

Same bound: $|\cot(\pi z)| \le \frac{e^{\pi}-1}{e^{\pi}+1}$

Let's illustrate solution of Ex 4a:

Consider same contour Γ_N and $f(z) = \frac{1}{z^2}$, $g(z) = \frac{\pi}{z^2}\cot(\pi z)$. Then, by the same argument as in the proof of Theorem 2, we see $\oint_{\Gamma_N} g(z)dz \xrightarrow[N\to\infty]{} 0$

$$\mathrm{Res}_0\, g(z) = \mathrm{Res}_0 \left(\frac{\pi}{z^2}\cdot\frac{1-\frac{1}{2!}(\pi z)^2+\frac{1}{4!}(\pi z)^4-\dots}{\pi z-\frac{1}{3!}(\pi z)^3+\frac{1}{5!}(\pi z)^5-\dots}\right)$$

$$= \mathrm{Res}_0\left(\frac{\pi}{z^2}\cdot\frac{1}{\pi z}\cdot\left(1-\frac{1}{2}\pi^2 z^2+\dots\right)\left(1+\frac{1}{6}\pi^2 z^2+\dots\right)\right) = \left(\frac{1}{6}-\frac{1}{2}\right)\pi^2 = -\pi^2/3.$$

By Cauchy Residue theorem:

$$\oint_{\Gamma_N} g(z)dz = \sum_{\substack{-N\le k\le N \\ k\ne 0}} \frac{1}{k^2} - \frac{\pi^2}{3} = 2\sum_{k=1}^{N}\frac{1}{k^2} - \frac{\pi^2}{3}$$

$\downarrow N \to \infty$

0

$$\Rightarrow \boxed{\sum_{k=1}^{\infty}\frac{1}{k^2} = \frac{\pi^2}{6}}$$