Lecture #1

MA 42500/52500 "Elements of Complex Analysis"

In school: $\mathbb{N} = \{1, 2, 3, \dots\}$

$\Downarrow$ to subtract bigger from smaller

$\mathbb{Z} = \{\dots, -2, -1, 0, 1, 2, \dots\}$

$\Downarrow$ to divide numbers

$\mathbb{Q} = \{\frac{a}{b} \mid a, b \in \mathbb{Z},\ b \neq 0\}$

$\Downarrow$

$\mathbb{R} = \{$all real numbers$\}$

$\Downarrow$

$\mathbb{C}$ ← main object for this course, which we shall introduce now.

Let i be the formal element, which is a solution of equation
$$x^2 = -1.$$

Def: A complex number is an expression of the form $z = a + bi$, where a, b - real numbers. Moreover, a is called the real part of $z = a+bi$ while b is called the imaginary part of z.

Notations: $\mathbb{C} = \{$the set of all complex numbers$\}$

$a = \mathrm{Re}(z)$, $b = \mathrm{Im}(z)$ for $z = a + bi$

Recall: On $\mathbb{Q}$ and $\mathbb{R}$ we had two basic operations "+", "·" subject to a bunch of relations, such as

$a + b = b + a$, $a + (b + c) = (a + b) + c$

$a \cdot b = b \cdot a$, $a \cdot (b \cdot c) = (a \cdot b) \cdot c$

$a \cdot (b + c) = a \cdot b + a \cdot c$

as well as the element $1, 0$ s.t.

$$a \cdot 1 = a = 1 \cdot a, \quad a + 0 = a = 0 + a$$

and any element $a \neq 0$ had inverse a^{-1} such that

$$a \cdot a^{-1} = 1 = a^{-1} \cdot a$$

Same is true on $\mathbb{C}$ as we shall see now.

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Addition: $\left.\begin{array}{l} z_1 = a_1 + b_1 i \\ z_2 = a_2 + b_2 i \end{array}\right\} \Rightarrow$ set $z_1 + z_2 = \boxed{(a_1+a_2) + (b_1+b_2)i}$

Multiplication: $z_1 \cdot z_2 = (a_1 + b_1 i)(a_2 + b_2 i) \xrightarrow[\text{\& use } i^2=-1]{\text{open brackets}} \boxed{(a_1 a_2 - b_1 b_2) + (a_1 b_2 + a_2 b_1) i}$

Home exercise: verify the first 5 properties from p. 1.

Q: What about division $\frac{z}{w}$?

Let's illustrate this first by computing z^{-1} for $z \neq 0$.

If $z = x + yi \Rightarrow \frac{1}{z} = \frac{1}{x+yi} = \frac{1}{x+yi} \cdot \underbrace{\frac{x-yi}{x-yi}}_{=1} = \frac{x-yi}{x^2+y^2} = \frac{x}{x^2+y^2} - \frac{y}{x^2+y^2} i$

So: $\boxed{(x+yi)^{-1} = \frac{x}{x^2+y^2} - \frac{y}{x^2+y^2} \cdot i}$

Ex: Write $\frac{-1+4i}{2+i}$ in the form $a + bi$

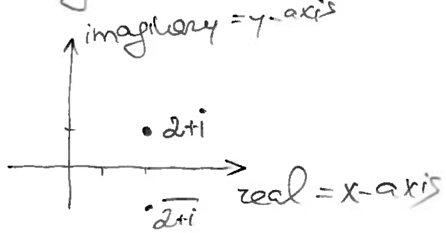
► $\frac{-1+4i}{2+i} = \frac{-1+4i}{2+i} \cdot \frac{2-i}{2-i} = \frac{2+9i}{5} = \frac{2}{5} + \frac{9}{5} i$ ■

Ex: Find i^{2025}.

► $i^2 = -1 \Rightarrow i^4 = 1 \Rightarrow i^{2024} = \underbrace{i^4 \cdot \ldots \cdot i^4}_{506} = 1 \Rightarrow i^{2025} = i^{2024} \cdot i = i$ ■

Q: How one can visualize $\mathbb{C}$?

Clearly, we can depict a complex number by a point in a plane



Note: Addition of complex numbers is then the same as usual addition of vectors!

Def: A *complex conjugate* of $z = a + b \cdot i$ is $\bar{z} = a - bi$

We already utilized this in division as $z \cdot \bar{z} = (a+bi)(a-bi) = \underbrace{a^2+b^2}_{\text{real!}}$

Geometrically, $\bar{z}$ is obtained from z by reflection in real axis.

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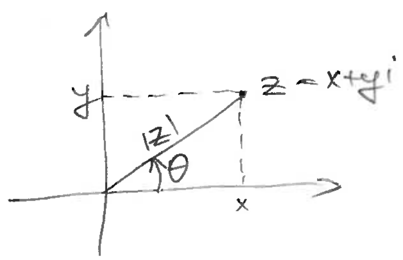
Q: What is the geometric interpretation of product?

Somehow, the formula

$$(x_1+y_1 i)(x_2+y_2 i)=(x_1x_2-y_1y_2)+(x_1y_2+x_2y_1)i$$

doesn't seem to have a simple meaning in usual coordinates

Hint: Switch to polar coordinates.



$$z=|z|\cdot\cos\theta+i\cdot|z|\sin\theta=|z|\cdot(\cos\theta+i\sin\theta)$$

Def: $|z|=\sqrt{x^2+y^2}$ = absolute value or modulus of z

As per the angle θ, note that adding 2π we actually get the same point on the plane.

Def: a) The principal value of the argument of $z\in\mathbb{C}$ is θ as in picture satisfying $-\pi<\theta\leq\pi$. This is denoted $\operatorname{Arg}(z)$.

b) The argument of z, denoted $\arg(z)$, is any angle of the form $\operatorname{Arg}(z)+2\pi k$ with $k\in\mathbb{Z}$.

Examples: $\operatorname{Arg}(-1)=\pi$, $\operatorname{Arg}(-1-i)=-\frac{3\pi}{4}$.

Note: $z\cdot\bar{z}=|z|^2$

Product revised

If $z_1=|z_1|\cdot(\cos\theta_1+i\cdot\sin\theta_1)$, $z_2=|z_2|\cdot(\cos\theta_2+i\sin\theta_2)$, then

$$z_1\cdot z_2=|z_1|\cdot|z_2|\cdot\Big(\underbrace{(\cos\theta_1\cdot\cos\theta_2-\sin\theta_1\cdot\sin\theta_2)}_{=\cos(\theta_1+\theta_2)}+\underbrace{(\cos\theta_1\cdot\sin\theta_2+\cos\theta_2\cdot\sin\theta_1)}_{=\sin(\theta_1+\theta_2)}i\Big)$$

$$=\boxed{|z_1||z_2|\cdot(\cos(\theta_1+\theta_2)+i\cdot\sin(\theta_1+\theta_2))}$$

So: Under the product, absolute values multiply, arguments add.

Warning: $\operatorname{Arg}(z_1\cdot z_2)\neq\operatorname{Arg}(z_1)+\operatorname{Arg}(z_2)$ as Right side may be $>\pi$.

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In this course we shall be mostly doing calculus on functions of a complex variable!

Let's conclude by a basic result:

Theorem: Any polynomial of degree n (in a variable z) has exactly n roots, if counted with multiplicities