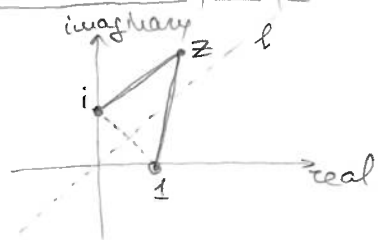
Lecture #2

Last time: Complex numbers, complex plane, polar coordinates.

Ex1: Describe all complex z such that $|z-i|=|z-1|$.

► Geometric proof



$|z-i|=$ distance b/w z and i

$|z-1|=$ distance b/w z and 1

So: We are looking for points on a plane s.t. distances from them to i and 1 are the same and those are easily seen to lie on the line ℓ above. Hence $z=\{x+x\cdot i \mid x\text{- real}\}$.

Algebraic Proof.

Write $z=x+yi \Rightarrow \begin{cases} z-i = x+(y-1)i \\ z-1=(x-1)+yi \end{cases} \Rightarrow \sqrt{x^2+(y-1)^2}=\sqrt{(x-1)^2+y^2}$

$\Rightarrow x^2+y^2-2y+1=x^2-2x+1+y^2 \Rightarrow x=y$. So $z=x+x\cdot i$ with any $x\in\mathbb{R}$ ◼

Last time to each $z=x+yi\in\mathbb{C}$ we associated $\mathrm{Re}(z)$, $\mathrm{Im}(z)$, $\bar{z}$.

Note: $\mathrm{Re}\, z=\dfrac{z+\bar{z}}{2}$

$\mathrm{Im}\, z=\dfrac{z-\bar{z}}{2i}$

$\bar{\bar{z}}=z$

Q1: Which of the following are true:

a) $\mathrm{Arg}(z_1\cdot z_2)=\mathrm{Arg}(z_1)+\mathrm{Arg}(z_2)$

b) $\arg(z_1\cdot z_2)=\arg(z_1)+\arg(z_2)$

Q2: Given a complex $z\neq 0$, describe $\mathrm{Arg}(\bar{z})$, $\mathrm{Arg}(z^{-1})$ via $\mathrm{Arg}(z)$. Similarly, describe $\arg(\bar{z})$, $\arg(z^{-1})$ via $\arg(z)$.

Warning: $\arg(z)$ is a multi-valued function, while $\mathrm{Arg}(z)$ is not continuous

↑ we will get to this next time

Lecture #2

Today: § 1.4–1.5

Question: Can one extend the "exponent" to a complex-valued function on a complex plane $e^z: \mathbb{C} \to \mathbb{C}$?

Basic Properties of Exponent:

$e^z \cdot e^w = e^{z+w}$, $e^0 = 1$, $(e^z)' = e^z$. ← want to keep all these properties

Let's now take $z = x + y \cdot i$ with $x, y \in \mathbb{R}$. Then first we have $e^z = e^x \cdot e^{iy}$.

Invoking the Taylor series expansion, we get:

$$e^{iy} = 1 + iy + \frac{(iy)^2}{2!} + \frac{(iy)^3}{3!} + \frac{(iy)^4}{4!} + \ldots = 1 + iy - \frac{y^2}{2!} - i \cdot \frac{y^3}{3!} + \frac{y^4}{4!} - \ldots$$

$$= \left(1 - \frac{y^2}{2!} + \frac{y^4}{4!} - \ldots\right) + \left(y - \frac{y^3}{3!} + \frac{y^5}{5!} - \ldots\right) i = \cos y + i \cdot \sin y$$

which results in Euler's equation:

$$\boxed{e^{i\theta} = \cos\theta + i \cdot \sin\theta \text{ for any } \theta \in \mathbb{R}}$$

In particular, for $\theta = \pi$, we get the famous formula $\boxed{e^{i\pi} = -1}$

Def: We define a complex exponent as follows: $\boxed{e^{x+yi} = e^x(\cos y + i\sin y)}$

At the moment, you can just think of $e^{i\theta}$ as a "shortcut" notation for $\cos\theta + i \cdot \sin\theta$. In particular, for $z \neq 0$, we get

$$\boxed{z = |z| \cdot e^{i \cdot \arg(z)}}$$, where $|z|$, $\arg(z)$ – polar coordinates

Hence, we can summarize the end of previous lecture as follows:

$$\text{if } z_1 = r_1 e^{i\theta_1},\ z_2 = r_2 e^{i\theta_2} \Rightarrow \begin{cases} z_1 \cdot z_2 = r_1 r_2 \cdot e^{i(\theta_1+\theta_2)} \\ z_1/z_2 = \frac{r_1}{r_2} \cdot e^{i(\theta_1-\theta_2)} \end{cases}$$

Also, evoking $\mathrm{Re}(e^{i\theta}) = \cos\theta$, $\mathrm{Im}(e^{i\theta}) = \sin\theta$, we thus get

$$\boxed{\cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2}, \quad \sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i}}$$

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Let us now prove the famous De Moivre's formula:

$$\boxed{(\cos\theta + i\sin\theta)^n = \cos(n\theta) + i\sin(n\theta)} \quad \text{for any } n \in \{1,2,3,\dots\}$$

▸ $\cos\theta + i\sin\theta = e^{i\theta}$, and $(e^{i\theta})^n = \underbrace{e^{i\theta}\cdot e^{i\theta}\cdot \ldots \cdot e^{i\theta}}_{n \text{ times}} = e^{i\cdot n\theta} = \cos(n\theta) + i\sin(n\theta)$ ◼

This allows to express $\cos(n\theta), \sin(n\theta)$ via $\cos\theta, \sin\theta$.

Ex2: Evaluate $\int_0^\pi \sin^4\theta \, d\theta$.

▸ This is actually a school-level problem, based on formula for $\sin^2\theta$. But for those who forgot the formula, let's derive it.

$$\sin^4\theta = \left(\frac{e^{i\theta}-e^{-i\theta}}{2i}\right)^4 = \frac{1}{16}(e^{i\theta}-e^{-i\theta})^4 = \frac{e^{i\cdot 4\theta} - 4\cdot e^{i\cdot 2\theta} + 6 - 4\cdot e^{i(-2\theta)} + e^{i(-4\theta)}}{16}$$

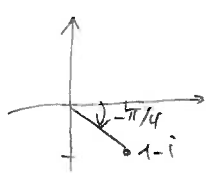
$$= \frac{2\cos(4\theta) - 8\cos(2\theta) + 6}{16}$$

$$\Rightarrow \int_0^\pi \sin^4\theta\,d\theta = \int_0^\pi \frac{1}{16}(2\cos(4\theta) - 8\cos(2\theta) + 6)\,d\theta = \frac{1}{16}\left\{\frac{1}{2}\sin(4\theta) - 4\sin(2\theta) + 6\theta\right\}\Big|_0^\pi$$

$$= \frac{3\pi}{8}$$ ◼

Ex3: Evaluate $(1-i)^{2025}$

▸ $1-i = \sqrt{2}\cdot e^{i(-\frac{\pi}{4})} \Rightarrow (1-i)^{2025} = \sqrt{2}^{2025}\cdot e^{i\cdot(-\frac{\pi}{4})\cdot 2025}$



But $-\frac{2025\pi}{4} = -506\pi - \frac{\pi}{4} = 2\pi\cdot(-253) - \frac{\pi}{4}$

$$\Rightarrow e^{i\cdot(-\frac{\pi}{4})\cdot 2025} = \cos\left(-\frac{\pi}{4}\right) + i\sin\left(-\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}i$$

So: $(1-i)^{2025} = \sqrt{2}^{2025}\cdot\frac{1-i}{\sqrt{2}} = \boxed{2^{1012} - i\cdot 2^{1012}}$ ◼

Note: If $z = re^{i\theta}$, then $z^{-1} = r^{-1}\cdot e^{i(-\theta)} \Rightarrow z^{-n} = (z^{-1})^n = r^{-n}\cdot e^{i(-n\theta)}$ for $n \in \{1,2,3,\dots\}$

so that formula $z^n = r^n\cdot\underbrace{(\cos(n\theta) + i\sin(n\theta))}_{e^{i\cdot n\theta}}$ holds for all $n \in \mathbb{Z}$.

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§1.5 (start)

Let's now learn how to extract n^{th} roots of complex numbers

Q: Solve $z^2=1$ $\quad z=\pm 1$

$z^4=1$ $\quad z=\pm 1, \pm i$

$z^n=1$ ⟵ let's write z in polar coordinates $z=re^{i\theta}$

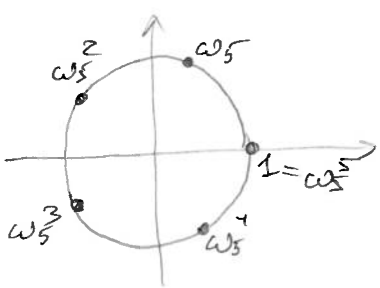
$$\Rightarrow z^n = r^n \cdot e^{in\theta} \Rightarrow z^n=1 \text{ iff } \begin{cases} r^n=1 \\ n\theta = 2\pi k, \ k\text{-integer} \end{cases}$$

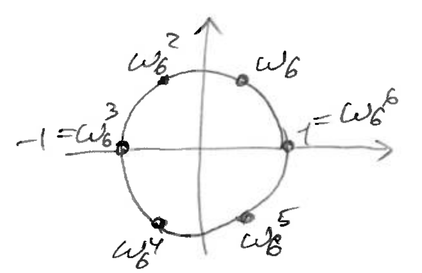
r – non-negative real, $r^n=1 \Rightarrow r=1$

$n\theta = 2\pi k \Rightarrow \theta = \frac{2\pi k}{n}$ BUT increasing k by n changes $\theta \rightsquigarrow \theta + 2\pi$ which corresponds to the same complex number

Thus: we get n roots of $z^n=1$ which correspond to vertices of regular n-gon inscribed in the unit circle.

E.g. $z^5=1$ has roots $1, \omega_5, \omega_5^2, \omega_5^3, \omega_5^4$ with $\boxed{\omega_n = \cos\left(\frac{2\pi}{n}\right) + i\sin\left(\frac{2\pi}{n}\right)}$





E.g. $z^6=1$ has roots $1, \omega_6, \omega_6^2, \omega_6^3, \omega_6^4, \omega_6^5$

Ex 4: Solve $z^4=3i$

$$\left.\begin{matrix} z=re^{i\theta} \Rightarrow z^4 = r^4 \cdot e^{i\cdot 4\theta} \\ 3i = 3\cdot e^{i\cdot\frac{\pi}{2}} \end{matrix}\right\} \Rightarrow \begin{cases} r^4=3 \\ 4\theta = \frac{\pi}{2} + 2\pi k, \ k\text{-integer} \end{cases} \Rightarrow \begin{cases} r = \sqrt[4]{3} \\ \theta = \frac{\pi}{8} + \frac{\pi k}{2}, \ k\text{-integer} \end{cases}$$

But: changing $k \rightsquigarrow k+4$ changes $\theta \rightsquigarrow \theta + 2\pi$ hence the same z.

So: $z = \sqrt[4]{3}\left(\cos\left(\frac{\pi}{8}+\frac{\pi k}{2}\right) + i\cdot\sin\left(\frac{\pi}{8}+\frac{\pi k}{2}\right)\right)$ with $k=-2,-1,0,1$ ∎