Lecture #3

We shall start today's lecture with practicing on Examples first.

Ex 1: Show that $-2+i$ is a solution of $z^2+4z+5=0$.

$$(-2+i)^2+4(-2+i)+5 = 4-4i+\underbrace{i^2}_{=-1}-8+4i+5=0$$

Note: We can actually find all solutions in the usual way:

$$0=z^2+4z+5=(z+2)^2+1 \Rightarrow (z+2)^2=-1 \Rightarrow z+2=\pm i \Rightarrow z=-2\pm i.$$

Ex 2: Find the principal argument and write in polar coordinates $\frac{1-i}{1+\sqrt{3}i}$

$$\left.\begin{array}{l} 1-i=\sqrt{2}\cdot e^{i(-\pi/4)} \\ 1+\sqrt{3}i=2\cdot e^{i\cdot\pi/3}\end{array}\right\} \Rightarrow \frac{1-i}{1+\sqrt{3}i}=\frac{1}{\sqrt{2}}e^{i\left(-\frac{\pi}{4}-\frac{\pi}{3}\right)}=\boxed{\frac{1}{\sqrt{2}}e^{i\left(-\frac{7\pi}{12}\right)}} \Rightarrow \operatorname{Arg}\left(\frac{1-i}{1+\sqrt{3}i}\right)=-\frac{7\pi}{12}$$

as it is in $(-\pi;\pi]$

Note: If you tried to write this as $a+bi$, you wouldn't see such answer:

$$\frac{1-i}{1+\sqrt{3}i}=\frac{1-i}{1+\sqrt{3}i}\cdot\frac{1-\sqrt{3}i}{1-\sqrt{3}i}=\frac{1-\sqrt{3}-(1+\sqrt{3})i}{4}$$

Ex 3: Write $-4\pi(1+\sqrt{3}i)$ in polar form $re^{i\theta}$.

$$\left.\begin{array}{l} r=|z|=\sqrt{(-4\pi)^2+(-4\pi\sqrt{3})^2}=8\pi \\ \cos\theta+i\sin\theta=\frac{z}{8\pi}=\frac{-4\pi(1+\sqrt{3}i)}{8\pi}=-\frac{1}{2}-\frac{\sqrt{3}}{2}i \Rightarrow \theta=\frac{4\pi}{3}\end{array}\right\} \boxed{8\pi e^{i\cdot\frac{4\pi}{3}}}$$

Note: If the question asked for the principal argument θ, then $\frac{4\pi}{3}$ wouldn't work as it is $>\pi$, and we would write $8\pi\cdot e^{i\left(-\frac{2\pi}{3}\right)}$

Ex 4: Express $\cos(4\theta)$ via $\cos\theta, \sin\theta$.

$$\left.\begin{array}{l} \cos(4\theta)=\operatorname{Re}\left((\cos\theta+i\sin\theta)^4\right) \text{ by De Moivre's Formula.} \\ \text{Recall: } (a+b)^4=a^4+4a^3b+6a^2b^2+4ab^3+b^4 \end{array}\right\} \Rightarrow$$

$$\Rightarrow \cos(4\theta)=\cos^4\theta-6\cdot\cos^2\theta\cdot\sin^2\theta+\sin^4\theta.$$

Note: One could further simplify using $\cos^2\theta+\sin^2\theta=1$.

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Ex5: Let $n \geq 2$ be integer and $\omega_n = e^{i \cdot \frac{2\pi}{n}} = \cos\left(\frac{2\pi}{n}\right) + i \sin\left(\frac{2\pi}{n}\right)$. Prove that

$$1 + \omega_n + \omega_n^2 + \_\_ + \omega_n^{n-1} = 0.$$

► Proof #1: By geometric progression formula:

$$1 + \omega_n + \ldots + \omega_n^{n-1} = \frac{1 - \omega_n^n}{1 - \omega_n} = \frac{1 - e^{i \cdot \frac{2\pi}{n} \cdot n}}{1 - \omega_n} = 0 \quad \checkmark$$

Proof #2 (equivalent to above): $(1-\omega_n)(1 + \omega_n + \omega_n^2 + \_\_ + \omega_n^{n-1}) =$

$$\begin{matrix} 1 + \omega_n + \omega_n^2 + \_\_ + \omega_n^{n-1} \\ - \omega_n - \omega_n^2 - \_\_ - \omega_n^{n-1} - \omega_n^n \end{matrix} = 1 - \omega_n^n = 0 \quad \checkmark$$

Geometric/Physics proof #3: The points $1, \omega_n, \omega_n^2, \_\_ \omega_n^{n-1}$ are vertices of a regular n-gon encircled by the unit circle. Hence, the sum is invariant under rotation by $\circlearrowleft \frac{2\pi}{n}$, hence is zero (i.e. center of mass is 0 due to the symmetry!) ∎

Ex6: For any integer $n \geq 2$, evaluate $1 + \cos\frac{2\pi}{n} + \cos\frac{4\pi}{n} + \_ + \cos\frac{2\pi(n-1)}{n}$.

► $\cos\frac{2\pi}{n} = \operatorname{Re} \omega_n$ and likewise $\cos\frac{2\pi \cdot k}{n} = \operatorname{Re}(\omega_n^k) \Rightarrow$

$$\Rightarrow 1 + \cos\frac{2\pi}{n} + \_\_ + \cos\frac{2\pi(n-1)}{n} = \operatorname{Re}(1 + \omega_n + \omega_n^2 + \_\_ + \omega_n^{n-1}) \overset{Ex5}{=\!=} \operatorname{Re}(0) = 0$$ ∎

(!) Note that quadratic equations can be solved the same way as in school. Namely, if $a, b, c \in \mathbb{C}$ and $a \neq 0$, then $az^2 + bz + c = 0$ can be written as $a\left(z + \frac{b}{2a}\right)^2 + c - \frac{b^2}{4a} = 0$, or as $\left(z + \frac{b}{2a}\right)^2 = \frac{b^2 - 4ac}{4a^2}$. Thus, solutions are:

$$\boxed{z = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}}$$

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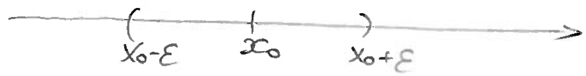
Today: §1.6.

Recall that in usual calculus, one studies $f:\mathbb{R}\to\mathbb{R}$. And for limits, continuity, one usually appeals to "ε-neighborhoods" of a point:

$(x_0-\varepsilon,\ x_0+\varepsilon)$ with $\varepsilon>0$



Now, when studying functions $f:\mathbb{C}\to\mathbb{C}$ of a complex variable it's quite natural to replace the above with

$$\boxed{D_\varepsilon(z_0) := \{z\in\mathbb{C} \mid |z-z_0|<\varepsilon\} = \text{open disk of radius } \varepsilon>0 \text{ centered at } z_0}$$

Note: $|z-z_0|$ = distance between z_0 & z.

Def: a) A point z_0 of a set S in $\mathbb{C}$ is an interior point if there is some $\varepsilon>0$ such that $D_\varepsilon(z_0)$ is in S, denoted $D_\varepsilon(z_0)\subseteq S$.

b) A set S in the complex plane $\mathbb{C}$ is open if each point is interior.

Examples:  - open,  - open,  - not open

Note: the value of ε in b) does depend on the point z_0!

Def: An open set S in $\mathbb{C}$ is (path) connected if any two points of S can be joined by a continuous curve in this set S.

Note: In fact, one can always make this curve to be piece-wise linear (= polygonal path)

Def: A set Ω in $\mathbb{C}$ is a domain if it is open AND (path connected)

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True / False:

a) If Ω_1, Ω_2 - open sets of $\mathbb{C}$, then so is their union $\Omega_1 \cup \Omega_2$

b) If Ω_1, Ω_2 - open sets of $\mathbb{C}$, then so is their intersection $\Omega_1 \cap \Omega_2$

c) If $\{\Omega_j\}_{j\geq 1}$ - open sets in $\mathbb{C}$, then so is their union $\bigcup_{j\geq 1} \Omega_j$

d) If $\{\Omega_j\}_{j\geq 1}$ - open sets in $\mathbb{C}$, then so is their intersection $\bigcap_{j\geq 1} \Omega_j$.

Def: a) A point z_0 of a set S in $\mathbb{C}$ is a boundary point if for any $\varepsilon>0$, the open disk $D_\varepsilon(z_0)$ contains a point from S and a point not in S.

b) A set S of $\mathbb{C}$ is closed if it contains all boundary points.