Lecture #4

- T/F: a) $\{\Omega_j\}_{j=1}^{\infty}$ - open sets in $\mathbb{C}$, then union $\bigcup_{j\geq 1} \Omega_j$ - open

  b) $\{\Omega_j\}_{j=1}^{\infty}$ - closed sets in $\mathbb{C}$, then intersection $\bigcap_{j\geq 1} \Omega_j$ - open.

a) True

Take any $z_0 \in \bigcup_{j\geq 1} \Omega_j$. Then $z_0 \in \Omega_k$ for some $k \geq 1$. Being open, Ω_k must contain $D_\varepsilon(z_0)$ for some $\varepsilon > 0$, hence, $D_\varepsilon(z_0) \subseteq \Omega_k \subseteq \bigcup_{j\geq 1} \Omega_j$.

b) False

Need to give just one counterexample. Consider $\Omega_k = \{z \in \mathbb{C} \mid |z| < \frac{1}{k}\}$, i.e. Ω_k - open disk centered at the origin of radius $\frac{1}{k}$ $\Rightarrow \Omega_k$ - open.
But $\bigcap_{j\geq 1} \Omega_j = \{0\}$ ← just a one point set, clearly not open!

- T/F: a) Ω_1, Ω_2 - connected $\Rightarrow \Omega_1 \cup \Omega_2$ - connected?

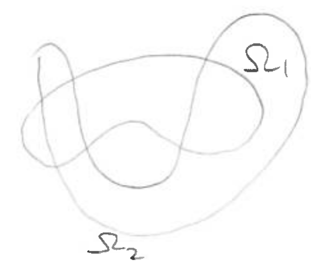
  b) Ω_1, Ω_2 - connected $\Rightarrow \Omega_1 \cap \Omega_2$ - connected?

a) False

Counterexample:

b) False

Counterexample



- Ex 1: Solve $z^4 + 5z^2 + 4 = 0$

If $y = z^2$, then we are first solving $\underbrace{y^2 + 5y + 4}_{=(y+1)(y+4)} = 0 \Rightarrow y = -1, -4$

$\Rightarrow z^2 = -1, -4 \Rightarrow z = \pm i, \pm 2i$

Note: The above roots split into pairs of complex conjugate ones! (which is always the case for polynomials with real coefficients)

Lecture #4 §2.1

This week we shall start discussing "calculus" for complex valued functions on a complex plane

$$f: \mathbb{C} \to \mathbb{C}$$

or more generally we shall consider $\mathbb{C}$-valued f defined on subsets of $\mathbb{C}$.

The terminology is the same as for real-valued functions:

- $f(z)$ is the image of z
- the set of all images is called the range of f
- the set on which f is defined is called the domain

Note: Unlike real-valued functions, it's quite impossible to draw a graph of $f: \mathbb{C} \to \mathbb{C}$. But one can still draw separately domain & range

Note that thinking of $\mathbb{C}$ as a real plane $\mathbb{R}^2$, one can (at the first try) interpret $f: \mathbb{C} \to \mathbb{C}$ as $\mathbb{R}^2 \to \mathbb{R}^2$, i.e. if $z = x+iy$ $(x,y \in \mathbb{R})$ and $f(z) = u(x,y) + i\cdot v(x,y)$

Example: $f(z) = z^2+4z+2$, then plugging $z = x+iy$, we get

$$f(x+iy) = (x+iy)^2 + 4(x+iy)+2 = x^2-y^2+2xy\cdot i + 4x + 4y\cdot i + 2$$

$$= \underbrace{(x^2-y^2+4x+2)}_{u(x,y)} + i\,\underbrace{(2xy+4y)}_{v(x,y)}, \text{ so that } \begin{array}{l} u(x,y) = x^2-y^2+4x+2 \\ v(x,y) = 2xy+4y \end{array}$$ ◻

Clearly, if you were given $u(x,y)$, $v(x,y)$ as above it may not be obvious to see if really corresponds to a $\mathbb{C}$-valued function on $\mathbb{C}$. However we shall soon give the answer to such question:

$\boxed{Q}$: Given two "good enough" functions $u(x,y)$, $v(x,y)$, is it possible to write $u(x,y)+i\cdot v(x,y)$ purely in terms of $z = x+iy$?

$\uparrow$ complex variable

Lecture #4 $\boxed{\S 2.2}$

Let us now discuss limits of sequences, functions, and continuity.

Def: A sequence of complex numbers $\{z_n\}_{n=1}^{\infty}$ has a limit $w \in \mathbb{C}$ if for any $\varepsilon > 0$ there is some integer N s.t. $|z_n - w| < \varepsilon$ for all $n > N$

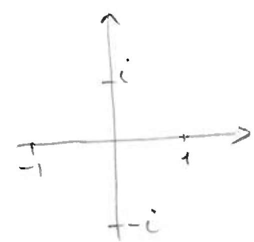
As in high-school, we are going to depict this by $\lim\limits_{n\to\infty} z_n = w$ or $z_n \xrightarrow[n\to\infty]{} w$.

Ex2: Find the limits of:

a) $\left(\frac{i}{2}\right)^n$, b) i^n, c) $\frac{10+i\cdot n^2}{2+2\cdot n^2}$

a) $\left|\left(\frac{i}{2}\right)^n\right| = \frac{1}{2^n} \xrightarrow[n\to\infty]{} 0$. So $\lim\limits_{n\to\infty} \left(\frac{i}{2}\right)^n = 0$.

b) $i^n = \begin{cases} 1 & \text{if } n \text{ is divisible by } 4 \\ i & \text{if } n \equiv 1 \text{ modulo } 4 \quad (\text{residue } 1 \bmod 4) \\ -1 & \text{if } n \equiv 2 \text{ modulo } 4 \\ -i & \text{if } n \equiv 3 \text{ modulo } 4 \end{cases}$



So clearly there is no limit!

c) $\frac{10+in^2}{2+2n^2} = \frac{i+\frac{10}{n^2}}{2+\frac{2}{n^2}} \xrightarrow[n\to\infty]{} \frac{i}{2}$

↑ Let's recall school argument:

$$\left|\frac{i+\frac{10}{n^2}}{2+\frac{2}{n^2}} - \frac{i}{2}\right| = \left|\frac{\cancel{2i}+\frac{20}{n^2}-\cancel{2i}-\frac{2i}{n^2}}{2\left(2+\frac{2}{n^2}\right)}\right| = \frac{\sqrt{404}}{n^2\cdot 2\cdot\left(2+\frac{2}{n^2}\right)} < \frac{\sqrt{404}}{4n^2} \xrightarrow[n\to\infty]{} 0$$

Clearly, all properties of limits from school still hold! In particular, if $z_n \xrightarrow[n\to\infty]{} a$, $w_n \xrightarrow[n\to\infty]{} b$, then:

- $z_n \pm w_n \xrightarrow[n\to\infty]{} a \pm b$
- $z_n \cdot w_n \xrightarrow[n\to\infty]{} a \cdot b$
- $\frac{z_n}{w_n} \xrightarrow[n\to\infty]{} \frac{a}{b}$ if $b \neq 0$.

(the proofs are the same as for real-valued sequences)

Lecture #4

Def: Let f be a function defined in some neighborhood of z_0, with a possible exception of z_0 (i.e. some $D_\varepsilon(z_0)\setminus\{z_0\}$). Then the limit of $f(z)$ as $z \to z_0$ is the number $w \in \mathbb{C}$ such that for any $\varepsilon>0$ there is $\delta>0$ with $|f(z)-w|<\varepsilon$ for any $0<|z-z_0|<\delta$

We shall write in the standard way: $\lim\limits_{z\to z_0} f(z)=w$, or $f(z)\xrightarrow[z\to z_0]{} w$.

Def: a) A function f defined in a neighborhood of z_0 is continuous at z_0 if $\lim\limits_{z\to z_0} f(z)=f(z_0)$

b) A function f is continuous on a set $S\subseteq\mathbb{C}$ if it is continuous at each $z_0\in S$.

Again, all basic properties for $f:\mathbb{R}\to\mathbb{R}$ still hold. In particular, if $\lim\limits_{z\to z_0} f(z)=a$, $\lim\limits_{z\to z_0} g(z)=b$, then:

- $\lim\limits_{z\to z_0}(f(z)\pm g(z))=a\pm b$
- $\lim\limits_{z\to z_0}(f(z)\cdot g(z))=a\cdot b$
- $\lim\limits_{z\to z_0}\dfrac{f(z)}{g(z)}=\dfrac{a}{b}$ if $b\neq 0$.

and as such if f,g are continuous at z_0, then so are: $f\pm g$, $f\cdot g$, and $\frac{f}{g}$ given $g(z_0)\neq 0$

(!) **Corollary**: Any polynomial $f(z)=a_nz^n+a_{n-1}z^{n-1}+\ldots+a_0$ is continuous at each $z_0\in\mathbb{C}$, and any rational function $\dfrac{a_nz^n+a_{n-1}z^{n-1}+\ldots+a_0}{b_nz^n+b_{n-1}z^{n-1}+\ldots+b_0}$ is continuous at $z_0\in\mathbb{C}$ s.t. z_0 is not a root of $b_nz^n+\ldots+b_0$

Ex 3: a) Compute $\lim\limits_{z\to i}\dfrac{z^2+3z}{z(z-2i)}$

b) Compute $\lim\limits_{z\to i}\dfrac{z^2+1}{z(z-i)}$.