Lecture #5

Ex 1: Verify that $w = f(z) = \frac{1}{z}$ maps:

a) circle $|z|=3$ to a circle $|w| = \frac{1}{3}$

b) circle $|z-1|=1$ to the vertical line $x = \frac{1}{2}$

▸ a) if $|z|=3$, then $|w| = \frac{1}{3}$ clearly. Backwards, for any $|w| = \frac{1}{3}$, if we set $z = w^{-1}$, then $|z|=3$ and $w = f(z)$.

Alternative way: $z = 3e^{i\theta} \Rightarrow w = \frac{1}{3}e^{i(-\theta)} \Rightarrow$ has range as claimed.

b) parametrize $z = 1 + \cos\theta + i\sin\theta$, $-\pi < \theta \leq \pi$. Then:

$$w = \frac{1}{1+\cos\theta + i\sin\theta} = \frac{1+\cos\theta - i\sin\theta}{(1+\cos\theta)^2 + (\sin\theta)^2} = \frac{(1+\cos\theta) - i\sin\theta}{2(1+\cos\theta)} = \frac{1}{2} - i\cdot\frac{\sin\theta}{1+\cos\theta}$$

So the image is indeed on a vertical line $x = \frac{1}{2}$.

To finish the proof we need to show $\frac{\sin\theta}{1+\cos\theta}$ can take any real value. By l'Hopital rule: $\lim\limits_{\theta\to\pi-0} \frac{\sin\theta}{1+\cos\theta} = +\infty$, $\lim\limits_{\theta\to-\pi+0} \frac{\sin\theta}{1+\cos\theta} = -\infty$, while the function $\frac{\sin\theta}{1+\cos\theta}$ is continuous on $(-\pi, \pi)$, hence, it achieves each value ∎

Note: Unlike for functions $f: \mathbb{R} \to \mathbb{R}$ in school where you could be often asked to draw a graph, this is not quite doable in the present setup of $f: \mathbb{C} \to \mathbb{C}$. However, we can still ask you to draw domain and range of f.

Ex 2: Let $z \in \mathbb{C}$ satisfy $|z| < 1$ and set $z_n = 1 + z + \_ + z^{n-1}$. Find $\lim\limits_{n\to\infty} z_n$.

▸ By geometric progression formula $z_n = \frac{1-z^n}{1-z}$

(if you forgot: $(1-z)(1+z+\_+z^{n-1}) = \begin{array}{l} 1+z+z^2+\_+z^{n-1} \\ -z-z^2-\ldots-z^{n-1}-z^n \end{array} = 1-z^n$)

We claim that $\lim\limits_{n\to\infty} \frac{1-z^n}{1-z} = \frac{1}{1-z}$. Indeed, we have

$$\left|\frac{1-z^n}{1-z} - \frac{1}{1-z}\right| = \frac{|z|^n}{|1-z|} \xrightarrow[n\to\infty]{} 0 \text{ as } |z|<1$$

∎

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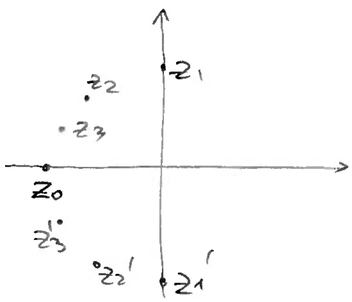
<u>Note</u>: If the limit $\lim_{z\to z_0} f(z)$ exists, then for any sequence of points $\{z_n\}_{n=1}^{\infty} \to z$ we should have the same value of $\lim_{n\to\infty} f(z_n)$.

<u>Ex3</u>: Show that $\operatorname{Arg} z$ is discontinuous at $\mathbb{R}_{<0}$.



If we take $z_n = |z_0| e^{i(\pi - \frac{\pi}{n})}$, then $z_n \to z_0$ and $\operatorname{Arg}(z_n) \to \pi$, while if we take $z_n' = |z_0| e^{-i(\pi - \frac{\pi}{n})}$, then $z_n' \to z_0$ and $\operatorname{Arg}(z_n') \to -\pi$

As $\pi \neq -\pi$, there is no $\lim_{z\to z_0} \operatorname{Arg} z$.

Let's now recall the very last example from last time:

- <u>Example</u>: $\lim_{z\to i} \underbrace{\frac{z^2+1}{z(z-i)}}_{f(z)}$

  Note that both numerator and denominator vanish at $z=i$, so we cannot just plug in. However, $z^2+1 = (z-i)(z+i)$, hence $\frac{z^2+1}{z(z-i)} = \frac{z+i}{z}$

  Hence $\lim_{z\to i} f(z) = \frac{2i}{i} = 2$. Note though that $f(z)$ is not defined at $z=i$!

<u>Def</u>: A function $f(z)$ has a <u>removable discontinuity</u> at z_0 if it becomes continuous at z_0 after redefining value $f(z_0)$

- Note that in contrast to $f: \mathbb{R} \to \mathbb{R}$ where $\lim_{x\to x_0} f(x)$ is the common value of right limit $\lim_{x\to x_0^+} f(x)$ and left limit $\lim_{x\to x_0^-} f(x)$, in the present setup we actually "approach $z_0 \in \mathbb{C}$ from any direction"

<u>Q</u>: $\lim_{z\to i} \frac{z^2+3z}{z^2+1} = ?$, $\lim_{z\to\infty} \frac{2z+2}{iz+4} = ?$, $\lim_{z\to\infty} \frac{2z^2+2}{iz+4} = ?$

To answer this question we first need to discuss what "∞" means.

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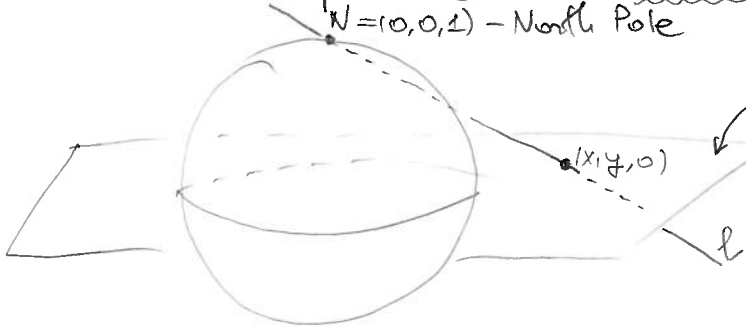
§1.7: Riemann sphere

Consider the sphere (called Riemann sphere) $x_1^2+x_2^2+x_3^2=1$



$\mathbb{C}$-plane $z=x+iy$ $\left(\begin{array}{l}\text{i.e. } x_3=0\\ x=x_1,\ y=x_2\end{array}\right)$

We have a natural bijection between points on the plane $x_3=0$ (which we think of as a complex plane) and points on the sphere excluding N, obtained by intersecting any line passing through N and a point in $\mathbb{C}$-plane with a sphere. Let's derive explicit formulas.

> **Lemma 1**: A point $z=x+iy\in\mathbb{C}=\{x_3=0 \text{ plane}\}$ corresponds to the point (x_1,x_2,x_3) on above sphere with $x_1=\frac{2\,\mathrm{Re}\,z}{|z|^2+1}$, $x_2=\frac{2\,\mathrm{Im}\,z}{|z|^2+1}$, $x_3=\frac{|z|^2-1}{|z|^2+1}$.

Any point on the line ℓ passing through $N=(0,0,1)$ and $(x,y,0)$ can be parametrized by $(x_1,x_2,x_3)=(0,0,1)+t\cdot(x,y,-1)$ with $t\in\mathbb{R}$, i.e.

$$x_1=tx,\quad x_2=ty,\quad x_3=1-t$$

For this point to lie on above sphere, need:

$$1=(tx)^2+(ty)^2+(1-t)^2=t^2(x^2+y^2+1)-2t+1$$

which has two solutions $t=0$ (corresponding to N=north pole) and $t=\frac{2}{x^2+y^2+1}=\frac{2}{|z|^2+1}$, where $z=x+iy$.

Plugging this into $x_1=tx$, $x_2=ty$, $x_3=1-t$ implies the claimed f-las. ▥

Example: For $z=1+i$, we get $(x_1,x_2,x_3)=\left(\frac{2}{3},\frac{2}{3},\frac{1}{3}\right)$

We can also derive the opposite formula: expressing x,y via x_1,x_2,x_3

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Lemma 2: A point (x_1, x_2, x_3) on a unit sphere has stereographic projection $z = x + iy$ with $x = \frac{x_1}{1-x_3}$, $y = \frac{x_2}{1-x_3}$

▶ As before: $x_1 = tx$, $x_2 = ty$, $x_3 = 1-t$

Solving $x_3 = 0 \Rightarrow t = 1 - x_3 \Rightarrow x = \frac{x_1}{1-x_3}$, $y = \frac{x_2}{1-x_3}$ ∎

As an important application of the above formula, we have:

Fact: All lines and circles in the complex plane correspond to circles on the Riemann sphere

▶ Indeed, any circle or line in $\mathbb{C}$-plane is given by

(1) $\boxed{A(x^2+y^2) + Bx + Cy + D = 0}$ with $A, B, C, D \in \mathbb{R}$, $A \geq 0$.

(Moreover $A = 0$ corresponds to lines, $A > 0$ - to circles)

Plugging above f-las $x = \frac{x_1}{1-x_3}$, $y = \frac{x_2}{1-x_3}$, and multiplying by $(1-x_3)^2$ get:

$$\boxed{A(\underbrace{x_1^2 + x_2^2}_{=1-x_3^2=(1-x_3)(1+x_3)}) + Bx_1(1-x_3) + Cx_2(1-x_3) + D(1-x_3)^2 = 0}$$

Dividing by $1-x_3$, we get an equation of a plane

(2) $\boxed{A(1+x_3) + Bx_1 + Cx_2 + D(1-x_3) = 0}$

Thus, the stereographic projection of (1) satisfies (2) and hence belongs to a circle obtained as the intersection of above plane and the unit sphere. It's clear that each point on that circle arises that way (reversing above argument) ∎

Note: Converseley every circle on the Riemann sphere is mapped under stereographic projection to a line or circle.

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By the above fact, an open disk around any point $z_0 \in \mathbb{C}$ will correspond to a cap around its stereographic projection (on the unit sphere) bounded by a circle, like in Fig1

Fig. 1

Fig. 2

However, we note a special role of the north pole "N": it doesn't correspond to any point on the complex plane. However, if we consider a small cap around N, like in Fig. 2, then under stereographic projection it clearly corresponds to $\{z \in \mathbb{C} \mid |z| \geq R\}$ for some real R.

(!) With this perspective in mind, we think of the Riemann sphere as a one-point compactification of the complex plane, where "N = North pole" corresponds to a point "∞". Thus $\mathbb{C} \cup \{\infty\} =: \hat{\mathbb{C}}$ is the extended complex plane

Let us now present the versions of limits, when either sequence tends to ∞, or the limit, or both.

Def: a) A sequence $\{z_n\}_{n=1}^{\infty}$ has a limit ∞, denoted $\lim_{n\to\infty} z_n = \infty$, if for any real R there is N such that $|z_n| > R$ for all $n > N$.

a') $\lim_{z\to z_0} f(z) = \infty$ if for any real R there is $\delta > 0$ s.t. $|f(z)| > R$ if $0 < |z - z_0| < \delta$

b) Likewise, we say $\lim_{z\to\infty} f(z) = w_0$ if for any $\varepsilon > 0$ there is $R \in \mathbb{R}$ such that $|f(z) - w_0| < \varepsilon$ whenever $|z| > R$.

c) Finally, we say $\lim_{z\to\infty} f(z) = \infty$ if for any $R \in \mathbb{R}$ there is $K \in \mathbb{R}$ s.t. $|f(z)| > R$ for $|z| > K$.

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Ex 4: Show that:

a) $\lim\limits_{z\to i} \dfrac{z^2+3z}{z^2+1} = \infty$

b) $\lim\limits_{z\to\infty} \dfrac{2z+2}{iz+4} = -2i$

c) $\lim\limits_{z\to\infty} \dfrac{2z^2+2}{iz+4} = \infty$

Warning: Unlike in school where you used two different symbols $+\infty, -\infty$ we now have a single "∞" when speaking about extended complex plane.