Lecture #6

Last week: limits & continuity of functions $f: \mathbb{C} \to \mathbb{C}$

Today: §2.3–2.4.

Let's start right away with the definition:

Def: Let f be $\mathbb{C}$-valued function defined in a neighborhood of z_0 (i.e. in some small open disk $D_\varepsilon(z_0)$). The <u>derivative</u> of f at z_0 is

$$\boxed{f'(z_0) = \frac{df}{dz}(z_0) := \lim_{\Delta z \to 0} \frac{f(z_0+\Delta z) - f(z_0)}{\Delta z}}$$

when z_0 is fixed we view $\frac{f(z_0+\Delta z)-f(z_0)}{\Delta z}$ as a function of Δz

Note: This looks exactly as the derivative of $\mathbb{R}$-valued function on $\mathbb{R}$, except that now $\Delta z \to 0$ means $|\Delta z| \to 0$ while it can approach "from any direction"

Example: a) If $f(z) = z$, then $f'(z_0) = \lim\limits_{\Delta z \to 0} \frac{(z_0+\Delta z) - z_0}{\Delta z} = 1$

b) If $f(z) = z^2$, then $f'(z_0) = \lim\limits_{\Delta z \to 0} \frac{(z_0+\Delta z)^2 - z_0^2}{\Delta z} = \lim\limits_{\Delta z \to 0} (2z_0 + \Delta z) = 2z_0$

c) More generally, if $f(z) = a_n z^n + a_{n-1} z^{n-1} + \ldots + a_1 z + a_0$ – any polynomial, then $f'(z_0) = n a_n \cdot z_0^{n-1} + (n-1) a_{n-1} z_0^{n-2} + \ldots + a_1$ (just like in school!)

Ex 1: Describe all points at which the following functions are differentiable:

a) $f: \mathbb{C} \to \mathbb{C}$ given by $f(z) = \bar{z}$

b) $f: \mathbb{C} \to \mathbb{C}$ given by $f(z) = \mathrm{Re}(z)$

c) $f: \mathbb{C} \to \mathbb{C}$ given by $f(z) = \mathrm{Im}(z)$

▸

a) $f'(z_0) = \lim\limits_{\Delta z \to 0} \frac{\overline{z_0+\Delta z} - \bar{z}_0}{\Delta z} = \lim\limits_{\Delta z \to 0} \frac{\overline{\Delta z}}{\Delta z}$

Let's write $\Delta z = |\Delta z| \cdot e^{i\theta}$. Then $\frac{\overline{\Delta z}}{\Delta z} = e^{i(-2\theta)}$ – can be any point on unit circle while $|\Delta z| \to 0$ $\Rightarrow$ no limit.

Alternatively, you can just note that if Δz approaches 0 along x-axis, then $\frac{\overline{\Delta z}}{\Delta z} \to 1$, while if Δz approaches 0 along y-axis, then $\frac{\overline{\Delta z}}{\Delta z} \to -1$. Therefore, the limit doesn't exist at any point.

b) Like in above line, if $\Delta z \to 0$ along x-axis, then $\frac{\mathrm{Re}(z_0+\Delta z) - \mathrm{Re}(z_0)}{\Delta z} \to 1$, while if $\Delta z \to 0$ along y-axis, then $\frac{\mathrm{Re}(z_0+\Delta z) - \mathrm{Re}(z_0)}{\Delta z} \to 0$. Hence nowhere differentiable.

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c) Similar argument!
So each of these 3 functions is actually nowhere differentiable! ∎

Following the same arguments as in usual calculus, we obtain:

Proposition: a) If $f(z), g(z)$ are differentiable at z_0, then:

$$(f+g)'(z_0) = f'(z_0) + g'(z_0)$$

$$(c\cdot f)'(z_0) = c\cdot f'(z_0) \quad \text{for any } c\in\mathbb{C}$$

$$(f\cdot g)'(z_0) = f'(z_0)\cdot g(z_0) + f(z_0)g'(z_0)$$

b) If $f(z), g(z)$ are differentiable at z_0 and $g(z_0)\neq 0$, then

$$\left(\frac{f}{g}\right)'(z_0) = \frac{f'(z_0)g(z_0) - f(z_0)g'(z_0)}{g(z_0)^2}$$

c) If g is differentiable at z_0 and f is differentiable at $g(z_0)$, then "chain rule": $\frac{d(f\circ g)}{dz}(z_0) = f'(g(z_0))\cdot g'(z_0)$

Ex2: Compute the derivative of $f(z) = \left(\frac{z^3+z+1}{z^2+4}\right)^{2025}$

$$f'(z_0) = 2025\cdot\left(\frac{z_0^3+z_0+1}{z_0^2+4}\right)^{2024}\cdot\frac{(3z_0^2+1)(z_0^2+4) - (z_0^3+z_0+1)\cdot 2z_0}{(z_0^2+4)^2}$$

∎

Q: What about complex exponent $z\mapsto e^z$: is it everywhere differentiable?

A: We shall now discuss a general criteria that will imply answer "Yes"

Terminology:

a) A $\mathbb{C}$-valued function $f(z)$ is called *analytic* on an open set Ω of a complex plane if $f'(z_0)$ exists for every $z_0\in\Omega$.

b) A function $f:\mathbb{C}\to\mathbb{C}$ is called *entire* if it is analytic on all $\mathbb{C}$.

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Theorem 1: If a function $f(z) = u(x,y) + i\cdot v(x,y)$ with $z = x+iy$ is differentiable at a point $z_0 = x_0 + iy_0$, then the following holds:

Cauchy-Riemann equations $\begin{cases} u_x(x_0,y_0) = v_y(x_0,y_0) \\ u_y(x_0,y_0) = -v_x(x_0,y_0) \end{cases}$

In particular, if $f(z)$ is analytic on an open set Ω, then these Cauchy-Riemann (CR) equations hold at each point of Ω.

Proof

Let's evaluate $\lim\limits_{\Delta z \to 0} \dfrac{f(z_0+\Delta z) - f(z_0)}{\Delta z}$ by first letting Δz approach 0 along real axis, i.e. we take $\Delta z = \varepsilon + 0\cdot i$ with $\varepsilon \to 0$. Then, we get:

$$f'(z_0) = \lim_{\varepsilon\to 0} \frac{u(x_0+\varepsilon, y_0) + i\cdot v(x_0+\varepsilon, y_0) - u(x_0,y_0) - i\cdot v(x_0,y_0)}{\varepsilon}$$

$$= \lim_{\varepsilon\to 0} \frac{u(x_0+\varepsilon,y_0) - u(x_0,y_0)}{\varepsilon} + i\cdot \lim_{\varepsilon\to 0} \frac{v(x_0+\varepsilon,y_0) - v(x_0,y_0)}{\varepsilon} = u_x(x_0,y_0) + i\cdot v_x(x_0,y_0)$$

Let's now evaluate above limit when $\Delta z = i\varepsilon$ with $\varepsilon \to 0$ being real:

$$f'(z_0) = \lim_{\varepsilon\to 0} \frac{u(x_0,y_0+\varepsilon) + i\cdot v(x_0,y_0+\varepsilon) - u(x_0,y_0) - i\, v(x_0,y_0)}{i\varepsilon}$$

$$= \frac{1}{i} \lim_{\varepsilon\to 0} \frac{u(x_0,y_0+\varepsilon) - u(x_0,y_0)}{\varepsilon} + \lim_{\varepsilon\to 0} \frac{v(x_0,y_0+\varepsilon) - v(x_0,y_0)}{\varepsilon}$$

$$= -i\cdot u_y(x_0,y_0) + v_y(x_0,y_0)$$

So: $u_x(x_0,y_0) + i\cdot v_x(x_0,y_0) = v_y(x_0,y_0) - i\cdot u_y(x_0,y_0)$ which implies the result ∎

To illustrate, let's consider $f(z) = \mathrm{Re}\, z$ from Ex 1(b). Then $\begin{cases} u(x,y) = x \\ v(x,y) = 0 \end{cases}$ and $u_x(x_0,y_0) = 1$ (for any $x_0, y_0 \in \mathbb{R}$) while $v_y(x_0,y_0) = 0 \neq 1$, hence by Thm 1 it's nowhere differentiable. Same can be applied to Ex 1(a,c).

Q: Is the CR equations also sufficient for f to be differentiable at $z_0 = x_0 + iy_0$?

A: Under some mild conditions, "Yes"!

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Theorem 2: Let $f(z) = u(x,y) + i\cdot v(x,y)$ with $z = x+iy$ be a function defined in some open neighborhood Ω of $z_0 = x_0 + iy_0$, i.e. in small disk $D_\varepsilon(z_0)$. Assume that that all four partial derivatives u_x, u_y, v_x, v_y exist and are continuous in this neighborhood Ω. Then if the Cauchy-Riemann equations hold at z_0, then $f'(z_0)$ exists!

In particular, if u_x, u_y, v_x, v_y are continuous on an open set Ω and satisfy Cauchy-Riemann equations at each point of Ω, then f - analytic in Ω

Terminology: If $u(x,y): \Omega \to \mathbb{R}$ is such that u_x, u_y exist and are continuous on Ω, then we say that u is C^1-smooth on Ω.

Example: Show that $f(z) = e^z$ is an entire function

If $z = x+iy$, then $u(x,y) = e^x \cdot \cos y$, $v(x,y) = e^x \cdot \sin y$

Clearly both u & v are C^1-smooth on all plane. Also:

$u_x = e^x \cos y = v_y$ & $u_y = -e^x \sin y = -v_x$ $\Rightarrow$ CR equations hold everywhere

Hence, by Thm 2, f is entire! ∎

Ex 3: Prove that if f is analytic in a domain Ω and $|f(z)|$ is constant in Ω, then actually $f(z)$ is constant in Ω.

[Think over at home and we shall discuss it next time]