Lecture #7

Due to the Survey' results, let's practice more today, leaving §2.5 to the end.

Ex1: (True/False) Any $f:\mathbb{C}\to\mathbb{C}$ differentiable at z_0 is continuous at z_0

▶ True: As $g(z):=\dfrac{f(z)-f(z_0)}{z-z_0}\xrightarrow[z\to z_0]{} a\in\mathbb{C} \Rightarrow f(z)-f(z_0)=a(z-z_0)+h(z)(z-z_0)$ with $h(z)\xrightarrow[z\to z_0]{}0$

Thus, $|f(z)-f(z_0)|\leq |a|\cdot|z-z_0|+|h(z)|\cdot|z-z_0|\longrightarrow 0$ as $z\to z_0$ ◻

Ex2: Compute $\lim\limits_{z\to i}\dfrac{1+z^6}{1+z^{10}}$

[Hint: As in usual calculus, we can apply L'Hopital rule which says that if $f(z), g(z)$ are analytic at z_0, with $f(z_0)=0=g(z_0)$, $g'(z_0)\neq 0$, then:

$$\lim_{z\to z_0}\frac{f(z)}{g(z)}=\lim_{z\to z_0}\frac{\frac{f(z)-f(z_0)}{z-z_0}}{\frac{g(z)-g(z_0)}{z-z_0}}=\frac{f'(z_0)}{g'(z_0)}$$]

▶ For $f(z)=1+z^6$, $g(z)=1+z^{10}$, note that $f(i)=0=g(i)$, while $f'(i)=6i^5$, $g'(i)=10i^9$

So: $\lim\limits_{z\to i}\dfrac{f(z)}{g(z)}=\dfrac{6i^5}{10i^9}=\dfrac{3}{5}$ as $i^4=1$ ◻

Ex3: At which points is $z\mapsto |z|^2=f(z)$ differentiable?

▶ Proof #1 (from definition)

$$f'(z_0)=\lim_{\Delta z\to 0}\frac{|z_0+\Delta z|^2-|z_0|^2}{\Delta z}=\lim_{\Delta z\to 0}\frac{(z_0+\Delta z)(\overline{z_0}+\overline{\Delta z})-z_0\cdot\overline{z_0}}{\Delta z}=\lim_{\Delta z\to 0}\frac{\overline{z_0}\cdot\Delta z+z_0\cdot\overline{\Delta z}+\Delta z\cdot\overline{\Delta z}}{\Delta z}$$

$$=\overline{z_0}+\lim_{\Delta z\to 0} z_0\cdot\frac{\overline{\Delta z}}{\Delta z}$$

If $z_0=0$, then we just get $\overline{z_0}=0 \Rightarrow f'(0)$ is well-defined and equals 0.

If $z_0\neq 0$, then if $f'(z_0)$ existed, we would also have $\lim\limits_{\Delta z\to 0}\dfrac{\overline{\Delta z}}{\Delta z}$ which we showed last time doesn't exist.

Answer: Only at 0.

Proof #2 (Cauchy-Riemann = CR) Just need to find point where $\begin{cases}u_x=v_y\\u_y=-v_x\end{cases}$

But $v(x,y)=0\Rightarrow v_x=0=v_y$, while $u(x,y)=x^2+y^2\Rightarrow u_x=2x$, $u_y=2y$

Hence CR holds only at $x=0, y=0\Rightarrow$ at $z=0$ ◻

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Ex 4: Is $f(z)=3x^2+2x-3y^2-1+(6xy+2y)i$ an entire function?

If so, express it solely via z.

- By definition $f(z)$ - entire $\Leftrightarrow$ f is differentiable at each point

For the latter it suffices to check CR at each point.

$$\left.\begin{array}{l} u(x,y)=3x^2+2x-3y^2-1 \Rightarrow u_x=6x+2,\ u_y=-6y \\ v(x,y)=6xy+2y \Rightarrow v_x=6y,\ v_y=6x+2 \end{array}\right\} \Rightarrow \text{CR holds!}$$

So: yes $f(z)$ is entire

- Looking carefully at $f(z)$ it is easy to recognize that $f(z)=3z^2+2z-1$ where $z=x+yi$

But: in general, if you didn't see this right away, just plug $x=\frac{z+\bar{z}}{2}$, $y=\frac{z-\bar{z}}{2i}$. Let's do it in above example:

$$f(z)=3\left(\frac{z+\bar{z}}{2}\right)^2+2\left(\frac{z+\bar{z}}{2}\right)-3\left(\frac{z-\bar{z}}{2i}\right)^2-1+\left(6\cdot\frac{z+\bar{z}}{2}\cdot\frac{z-\bar{z}}{2i}+2\cdot\frac{z-\bar{z}}{2i}\right)i$$

$$=\frac{3}{4}(z^2+\bar{z}^2+2z\bar{z})+(z+\bar{z})+\frac{3}{4}(z^2+\bar{z}^2-2z\bar{z})-1+\frac{6}{4}(z+\bar{z})(z-\bar{z})+(z-\bar{z})$$

$$=\frac{6}{4}z^2+\frac{6}{4}\bar{z}^2+z+\bar{z}-1+\frac{6}{4}z^2-\frac{6}{4}\bar{z}^2+z-\bar{z}=3z^2+2z-1$$ ∎

Note: Once you know that $f(z)$ is entire and plug $x=\frac{z+\bar{z}}{2}$, $y=\frac{z-\bar{z}}{2i}$, you are guaranteed that after all cancelations there will be no $\bar{z}$!

! Comment regarding some Hwk 2 problems from §1.7, 2.1 that were asking to geometrically describe $w=f(z)$ as a transformation of the Riemann sphere:

- if you don't know geometric answer, there is only one way to do:

$$\left.\begin{array}{l} z \mapsto (x_1,x_2,x_3) \text{ via Lemma 1 of Lecture 5} \\ w \mapsto (x_1',x_2',x_3') \quad -//- \end{array}\right\} \text{try to relate } (x_1',x_2',x_3') \text{ to } (x_1,x_2,x_3)$$

- In #15 from p. 58 it will actually be much faster to do opposite way:

geom. description

$$\left.\begin{array}{l} (x_1,x_2,x_3) \mapsto z \text{ via Lemma 2 of Lecture 5} \\ (x_1',x_2',x_3') \mapsto w \quad -//- \end{array}\right\} \text{verify these } z,w \text{ are related by stated f-la}$$

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The essence of Cauchy-Riemann eqs from [Thm 1, Thm 2 of Lecture 6] is as follows. Given $u, v: \mathbb{R}^2 \to \mathbb{R}$ such that u_x, u_y, v_x, v_y exist and are continuous:

$$\boxed{\begin{array}{l} f(z=x+iy) := u(x,y) + i\cdot v(x,y) \\ \text{is differentiable at } z_0 = x_0+iy_0 \end{array} \iff \underline{\underline{CR}} \text{ holds } \begin{cases} u_x(x_0,y_0) = v_y(x_0,y_0) \\ u_y(x_0,y_0) = -v_x(x_0,y_0) \end{cases}}$$

[<u>Note</u>: The matrix consisting of first partial derivatives now takes the form

$$\begin{pmatrix} u_x & u_y \\ v_x & v_y \end{pmatrix} \underset{=}{\overset{CR}{=}} \begin{pmatrix} A & B \\ -B & A \end{pmatrix}$$

which corresponds to a linear map $\mathbb{R}^2 \to \mathbb{R}^2$ which is a composition of rotation and scaling

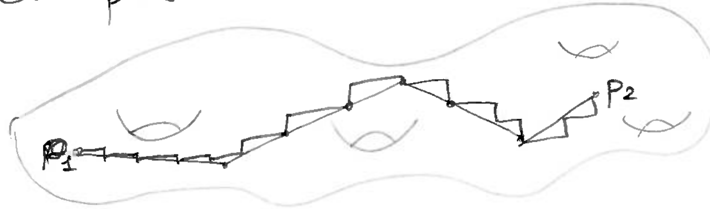
which corresponds to $(A+Bi)\cdot$ as a map $\mathbb{C} \to \mathbb{C}$

Lemma: If $f(z)$ - analytic in a domain Ω, $f'(z)=0$ in $\Omega \Rightarrow f(z) = \text{const}$

▸ The equality $f'(z_0) = u_x(x_0,y_0) + i\cdot v_x(x_0,y_0) = -i\cdot u_y(x_0,y_0) + v_y(x_0,y_0)$ from Lecture 6 imply:

$u_x = 0 = u_y$ in Ω and similarly $v_x = 0 = v_y$ in Ω

As Ω is connected, any two points can be connected by a piece-wise linear path in Ω, which can be further approximated by a piece-wise linear path with each segment being horizontal or vertical.



$$\left.\begin{array}{l} \text{As } u_x = 0 \Rightarrow u \text{ is constant on all horizontal segments} \\ \quad u_y = 0 \Rightarrow u \quad \text{—"—} \quad \text{vertical segments} \end{array}\right\} \Rightarrow u(p_1) = u(p_2)$$

Similarly: $v(p_1) = v(p_2) \Rightarrow f$ is constant ∎

Corollary: a) If f, g - analytic in domain and $f' = g'$ in $\Omega \Rightarrow f - g = \text{constant}$

b) If f - analytic and $\mathrm{Im}\, f$ is constant $\Rightarrow f$ - constant

c) If f - analytic and $\mathrm{Re}\, f$ is constant $\Rightarrow f$ - constant

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Ex 5: Show that if $f(z)$ is analytic in domain and $|f(z)|=\text{constant}$, then actually $f(z)=\text{constant}$.

▸ • If $|f(z)|=0 \Rightarrow$ clear

• Assume $|f(z)|=c>0$. Write $f(x+iy)=u(x,y)+i\cdot v(x,y)$.

Then $u^2(x,y)+v^2(x,y)=c^2$. Apply ∂_x or ∂_y to get:

$$\begin{cases} 2\cdot u\cdot u_x + 2v\cdot v_x = 0 \\ 2\cdot u\cdot u_y + 2v\cdot v_y = 0 \end{cases}$$

← each u, v, u_x, u_y, v_x, v_y is evaluated at the given point

But f - analytic $\overset{CR}{\Longrightarrow} u_x=v_y,\ u_y=-v_x$. Hence, we get:

$$\left.\begin{array}{l} \begin{cases} u\cdot v_y + v\cdot v_x = 0 \\ -u\cdot v_x + v\cdot v_y = 0 \end{cases} \Rightarrow \begin{pmatrix} v & u \\ -u & v \end{pmatrix}\cdot\begin{pmatrix} v_x \\ v_y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \\ \text{However, } \det\begin{pmatrix} v & u \\ -u & v \end{pmatrix} = u^2+v^2 = c^2 \neq 0 \end{array}\right\} \Rightarrow v_x=v_y=0 \overset{CR}{\Rightarrow} u_x=u_y=0.$$

Hence, by the same argument as in the proof of lemma above:

$u=\text{const},\ v=\text{const} \Rightarrow f - \text{constant}$

§2.5 Harmonic functions

Def: A function $g: \underset{\text{domain in } \mathbb{R}^2}{\Omega} \to \mathbb{R}$ is called harmonic if it satisfies the following equation $\underbrace{\frac{\partial^2 g}{\partial x^2} + \frac{\partial^2 g}{\partial y^2} = 0}_{\text{Laplace's equation}}$

Laplace's equation, often written $\nabla^2 g = 0$ where $\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$.

Theorem: If $f: \mathbb{C} \to \mathbb{C}$ is analytic in a domain $\Omega \subseteq \mathbb{C}$ then $\operatorname{Re}(f), \operatorname{Im}(f)$ are both harmonic functions in Ω

▸ Write $f(x+iy)=u(x,y)+i\cdot v(x,y)$. We shall see later: $u(x,y), v(x,y)$ admit continuous partial derivatives of any order!

(this is a surprising fact, very different from usual calculus)

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Continuation of the proof

CR : $u_x = v_y \xrightarrow{\text{apply } \partial_x} u_{xx} = v_{xy}$

$u_y = -v_x \xrightarrow[\text{apply } \partial_y]{} u_{yy} = -v_{yx}$

$\Rightarrow u_{xx} + u_{yy} = v_{xy} - v_{yx} = 0$ as both v_{xy}, v_{yx} are continuous

$\Downarrow$

$\boxed{u - \text{harmonic}}$

CR : $u_x = v_y \xrightarrow{\text{apply } \partial_y} u_{yx} = v_{yy}$

$u_y = -v_x \xrightarrow[\text{apply } \partial_x]{} u_{xy} = -v_{xx}$

$\Rightarrow v_{xx} + v_{yy} = u_{yx} - u_{xy} = 0$

$\Downarrow$

$\boxed{v - \text{harmonic}}$

$\blacksquare$

Next time: finish § 2.5