Lecture #8

Last time we ended with the following result:

Theorem 1: If $f(z=x+iy)=u(x,y)+i\cdot v(x,y)$ is an analytic function in a domain Ω then both $u,v:\Omega\to\mathbb{R}$ are harmonic

Note: One can rephrase this as follows: if $u,v:\Omega\to\mathbb{R}$ are C^2-smooth and satisfy CR in Ω, then u,v are harmonic in Ω.

In fact, while last time we saw one needs this C^2-smoothness, it follows from:

Surprising Fact: If $f:\Omega\to\mathbb{C}$ is analytic, then $f':\mathbb{C}\to\mathbb{C}$ and all $f^{(n)}:\mathbb{C}\to\mathbb{C}$ are

Q: Given a C^2-smooth harmonic $u:\Omega\to\mathbb{R}$, can we find analytic $f:\Omega\to\mathbb{C}$ with $Re(f)=u$, i.e. can we find another harmonic $v:\Omega\to\mathbb{R}$ s.t. (u,v)-CR

Let's start with an example.

Ex 1: Let $u(x,y)=x^2-y^2+3xy+1$. Verify it is harmonic. Find all $v(x,y)$ such that u,v satisfy CR. Verify v is harmonic

- $u_{xx}=2,\ u_{yy}=-2 \Rightarrow u_{xx}+u_{yy}=0 \Rightarrow u$ - harmonic.
- Look for $v(x,y)$ such that $\begin{cases} v_x=-u_y=2y-3x \\ v_y=u_x=2x+3y. \end{cases}$

Step 1: Solve $v_x=2y-3x$ for any fixed y: $v(x,y)=\int(2y-3x)dx=2xy-\frac{3}{2}x^2+A(y)$

Step 2: Solve $2x+3y=v_y=2x+A'(y) \Rightarrow A(y)=\frac{3}{2}y^2+\underbrace{B}_{\text{constant}}$

$\Rightarrow \boxed{v(x,y)=2xy-\frac{3}{2}x^2+\frac{3}{2}y^2+B}$

- $v_{xx}=-3,\ v_{yy}=3 \Rightarrow v$ - harmonic

Note: It is clear that if $v,\tilde{v}$ - two possible solutions, then $\partial_x(v-\tilde{v})=0=\partial_y(v-\tilde{v})$ hence $v,\tilde{v}$ differ by adding a constant (given Ω - domain)

Rmk: In Step 2 there will never be x after cancelations unless you made error

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In fact, the general result from calculus asserts that if Ω is an open disk, a rectangle, or more generally a domain without "holes", then $\begin{cases} v_x = \varphi(x,y) \\ v_y = \psi(x,y) \end{cases}$ admits a solution iff $\varphi_y = \psi_x$. But in our case of $\varphi = -u_y$, $\psi = u_x$ (by CR), this reduces to $u_{xx} + u_{yy} = 0$ which holds as u-harmonic!

$\Downarrow$

Theorem 2: On any domain without "holes" (known as simply-connected) for any harmonic u there is another harmonic v such that u, v satisfy CR and thus give rise to analytic $f(z) = u + iv$

Def: Such v is called harmonic conjugate of u (and is unique up to adding a constant)

A classical counterexample to such statement for not simply-connected domains is (we shall return to it in a few lectures):

Ex2*: Verify that $u(x,y) = \ln\sqrt{x^2+y^2} : \mathbb{R}^2 \setminus \{0\} \to \mathbb{R}$ is harmonic, BUT there is no analytic $f(z) : \mathbb{C} \setminus \{0\} \to \mathbb{C}$ with $\mathrm{Re}(f) = u$.

Let us now solve the following exercise (a bit tricky!):

Ex3: Verify that $u(x,y) = r^n \cos(n\theta)$, where n-integer and r, θ-polar coor, is a harmonic function, and find its harmonic conjugate

One can recall that $r^n \cos(n\theta) = \mathrm{Re}(z^n)$ with $z = re^{i\theta}$ by De Moivre's thm. But $z \mapsto z^n$ is analytic on $\mathbb{C} \setminus \{0\}$. Hence, $\mathrm{Re}(z^n)$ - harmonic and moreover $\mathrm{Im}(z^n) = r^n \sin(n\theta)$ is a harmonic conjugate. $\blacksquare$

Rmk: In the next homework you will see that Laplace's equation in polar coordinates takes the form $\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\cdot\frac{\partial}{\partial r} + \frac{1}{r^2}\cdot\frac{\partial^2}{\partial \theta^2}\right)u = 0$. Hence, the first part of Ex 3 can be easily verified directly

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We shall conclude §2.5 by discussing level curves $u(x,y)=A\in\mathbb{R}$ and $v(x,y)=B\in\mathbb{R}$. Recall that the gradient vector field $\nabla u=\begin{pmatrix}u_x\\u_y\end{pmatrix}$ is orthogonal to the level curves of $u:\Omega\to\mathbb{R}$. Likewise, $\nabla v=\begin{pmatrix}v_x\\v_y\end{pmatrix}$ is orthogonal to the level curves of $v:\Omega\to\mathbb{R}$. But

$$\nabla u\cdot\nabla v=u_xv_x+u_yv_y\overset{CR}{=\!=}-v_yv_x+v_xv_y=0\Rightarrow\nabla u\perp\nabla v \text{ at each point}$$

$$\Downarrow$$

Fact: Level curves of u and v are always orthogonal

§3.1 Polynomials and rational functions

Recall that polynomials in 1 variable are $p(z)=a_nz^n+a_{n-1}z^{n-1}+\ldots+a_1z+a_0$. If $a_n\neq0$, then degree of $p(z)$ is $\deg(p)=n$. As stated in 1^{st} class:

Theorem 3 (Fundamental Theorem of Algebra): Any non-constant polynomial with complex coefficients has at least one root in $\mathbb{C}$. Therefore, each polynomial can be factorized as $p(z)=a_n(z-z_1)\ldots(z-z_n)$ for some $z_1,z_2,\ldots,z_n\in\mathbb{C}$

Ex 4: Prove that any nonconstant polynomial with real coefficients can be factored into product of linear and quadratic polynom-s with real coefficients.

▶ By Hwk #1: if $p(z_0)=0\Rightarrow p(\overline{z_0})=0$. Thus, if $z_0\in\mathbb{C}$ - is a root of $p(z)$ which is not real, then $\overline{z_0}$ - also a root. However:

$(z-z_0)(z-\overline{z_0})=z^2-2\,\mathrm{Re}(z_0)\cdot z+|z_0|^2$ which has real coeff-s.

The result follows! ◼

Note: If some $z_0\in\mathbb{C}$ appears k times among $\{z_1,\ldots,z_n\}$ in Thm 3, then $k=$ multiplicity of root z_0.

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Ex 5: Write $1+z+z^2+z^3+z^4+z^5+z^6$ in the factored form (as a product of linear factors)

$$1+z+\dots+z^6 = \frac{1-z^7}{1-z} = \frac{z^7-1}{z-1}$$

By Lecture 2: $z^7-1=(z-1)(z-\omega_7)(z-\omega_7^2)(z-\omega_7^3)(z-\omega_7^4)(z-\omega_7^5)(z-\omega_7^6)$, where $\omega_7 = e^{i\cdot\frac{2\pi}{7}}$. Hence:

$$1+z+\dots+z^6 = (z-\omega_7)(z-\omega_7^2)(z-\omega_7^3)(z-\omega_7^4)(z-\omega_7^5)(z-\omega_7^6)$$

To motivate the following construction (to be carried out next time) let's recall how to calculate $\int \frac{x-2}{x(x-3)}dx$. Idea: partial fraction decomposition, i.e. look for constants A, B such that $\frac{x-2}{x(x-3)} = \frac{A}{x} + \frac{B}{x-3}$. Bringing to common denominator, get: $(A+B)x - 3A = x-2 \Rightarrow \begin{cases} A+B=1 \\ -3A=-2 \end{cases} \Rightarrow \begin{cases} A = 2/3 \\ B = 1/3 \end{cases}$

Hence, $\int \frac{x-2}{x(x-3)}dx = \frac{2}{3}\ln|x| + \frac{1}{3}\ln|x-3| + C$

Next time: Discuss this process for any rational function.