Lecture #9 (§3.1–3.2)

Ex1: Is $|z|^2+e^{z^2}$ an entire function?

▸ No. If $f(z)=|z|^2+e^{z^2}$ was entire, then $|z|^2 = f(z)-e^{z^2}$ would also be. But $|z|^2 = z\cdot\bar{z}$, and so $\bar{z} = \frac{f(z)-e^{z^2}}{z}$ would be analytic on $\mathbb{C}\setminus\{0\}$. But we know that $\bar{z}$ is nowhere differentiable, a contradiction ∎

Ex2: Find poles and their multiplicities of $R(z) = \dfrac{z^4-3z^2+2}{(z-1)(z^2+4)^2}$.

▸ Note that
$$\left.\begin{aligned} z^4-3z^2+2 &= (z^2-1)(z^2-2) = (z-1)(z+1)(z-\sqrt{2})(z+\sqrt{2}) \\ (z-1)(z^2+4)^2 &= (z-1)(z-2i)^2(z+2i)^2 \end{aligned}\right\} \Rightarrow$$

$\Rightarrow R(z) = \dfrac{(z+1)(z-\sqrt{2})(z+\sqrt{2})}{(z-2i)^2(z+2i)^2}$ has poles at $\pm 2i$, both of multiplicity 2 ∎

Ex3: Decompose z^2+3z+2 in terms of powers of $z+1$.

▸ __Solution 1__: Want $z^2+3z+2 = a_n(z+1)^n + a_{n-1}(z+1)^{n-1} + \_\_ + a_1(z+1)+a_0$, $a_n\neq 0$

Comparing max powers of z we see that $n=2$ & $a_n=1$. Thus:

$z^2+3z+2 = (z+1)^2 + a_1(z+1)+a_0 \Rightarrow z+1 = a_1(z+1)+a_0 \Rightarrow a_1 = 1,\ a_0=0$. So:

$$z^2+3z+2 = 1\cdot(z+1)^2 + 1\cdot(z+1)$$

__Note__: The fact $a_0=0$ actually implies that -1 is a root!

__Solution 2__: If we set $w:=z+1$, then $z=w-1$. Thus, we get answer by plugging $w-1$ instead of z into original polynomial:

$$z^2+3z+2 = (w-1)^2+3(w-1)+2 = w^2+w \underset{w=z+1}{=} (z+1)^2+(z+1)$$

__Solution 3__: Finally, one could also apply the Taylor form expansion

If $p(z) = a_0+a_1(z-z_0)+a_2(z-z_0)^2+\_\_+a_n(z-z_0)^n$, then

$$\left.\begin{aligned} a_0 &= p(z_0) \\ 1!\cdot a_1 &= p'(z_0) \\ 2!\cdot a_2 &= p''(z_0) \\ &\vdots \\ n!\, a_n &= p^{(n)}(z_0) \end{aligned}\right\} \Rightarrow \boxed{a_k = \frac{1}{k!}p^{(k)}(z_0)} \quad \text{with } k! = 1\cdot 2\cdot \_\_ \cdot k$$

∎

Lecture #9

Note: For polynomial $p(z)$ of degree n, the Taylor series truncates, i.e. has finite number of nonzero terms as $p^{(>n)}(z)=0$.

Let's now switch to rational functions and reconsider the example from the end of Lecture 8, where we decomposed $\frac{x-2}{x(x-3)}$ into partial fractions

$$\frac{x-2}{x(x-3)} = \frac{A}{x} + \frac{B}{x-3}$$

Approach 1 (usually taught in school/calculus class): reduce to linear eqs

$$x-2 = A(x-3) + B\cdot x \iff \begin{cases} A+B=1 \\ -3A=-2 \end{cases}$$

Approach 2: Find constants A, B by specializing x!

Plug $x=0$ into $x-2 = A(x-3)+Bx$ to find $-2=-3A \Rightarrow A=2/3$

Plug $x=3$ — " — to find $1=3B \Rightarrow B=1/3$

Ex 4. Decompose $\frac{2z+i}{z^3+z}$ into partial fractions.

► $z^3+z = z(z-i)(z+i)$, hence, we are looking for decomposition

$$\frac{2z+i}{z^3+z} = \frac{A}{z} + \frac{B}{z-i} + \frac{C}{z+i} \qquad \text{goal: find } A, B, C$$

Bringing to common denominator find:

$$2z+i = A(z-i)(z+i) + Bz(z+i) + Cz(z-i)$$

Plug $z=0$: $i=A$

Plug $z=i$: $3i = B\cdot i\cdot 2i = -2B \Rightarrow B = -\frac{3i}{2}$

Plug $z=-i$: $-i = C\cdot(-i)(-2i) = -2C \Rightarrow C=\frac{i}{2}$

$$\boxed{\frac{i}{z} + \frac{-\frac{3i}{2}}{z-i} + \frac{\frac{i}{2}}{z+i}}$$

Ex 4': Same for $\frac{2z^4+2z^2+2z+i}{z^3+z}$

► Note $2z^4+2z^2+2z+i = 2z\cdot(z^3+z)+2z+i \overset{\text{Ex 4}}{\Longrightarrow} \boxed{2z + \frac{i}{z} + \frac{-\frac{3i}{2}}{z-i} + \frac{\frac{i}{2}}{z+i}}$

Lecture #9

The above examples illustrate two key general ideas:

1) if denominator has only multiplicity 1 roots, then all coefficients in partial fraction decomposition are obtained by evaluation at these roots (after bringing to the common denominator)

2) if $R(z) = \frac{p(z)}{q(z)}$ and $\deg p \geq \deg q$, then you first write $p(z) = q(z) \cdot a(z) + \bar{p}(z)$ with $\deg \bar{p} < \deg q$

Theorem: If $R(z) = \dfrac{p(z)}{b_n (z-\zeta_1)^{d_1} \cdots (z-\zeta_r)^{d_r}}$ is a rational function with the denominator $q(z) = b_n (z-\zeta_1)^{d_1} \cdots (z-\zeta_r)^{d_r}$ having roots $\zeta_1, \dots, \zeta_r$ of multiplicities $d_1, \dots, d_r$, and we assume $\deg p < \deg q = d_1 + \dots + d_r$, then $R(z)$ admits the following partial fraction decomposition:

$$R(z) = \frac{A_0^{(1)}}{(z-\zeta_1)^{d_1}} + \frac{A_1^{(1)}}{(z-\zeta_1)^{d_1-1}} + \dots + \frac{A_{d_1-1}^{(1)}}{(z-\zeta_1)}$$
$$+ \frac{A_0^{(2)}}{(z-\zeta_2)^{d_2}} + \frac{A_1^{(2)}}{(z-\zeta_2)^{d_2-1}} + \dots + \frac{A_{d_2-1}^{(2)}}{(z-\zeta_2)}$$
$$+ \dots + \frac{A_{d_r-1}^{(r)}}{(z-\zeta_r)}$$

with $\boxed{A_s^{(j)} = \lim_{z \to \zeta_j} \frac{1}{s!} \frac{d^s}{dz^s} \left((z-\zeta_j)^{d_j} \cdot R(z) \right)}$

↑ Read the proof in p. 107 of your textbook.

Note: 1) $\dfrac{A_0^{(j)}}{(z-\zeta_j)^{d_j}} + \dots + \dfrac{A_{d_j-1}^{(j)}}{(z-\zeta_j)} = \dfrac{A_0^{(j)} + A_1^{(j)}(z-\zeta_j) + \dots + A_{d_j-1}^{(j)} (z-\zeta_j)^{d_j-1}}{(z-\zeta_j)^{d_j}}$

2) if $\deg p \geq d_1 + \dots + d_r$, we first apply division algorithm as in 2) above.

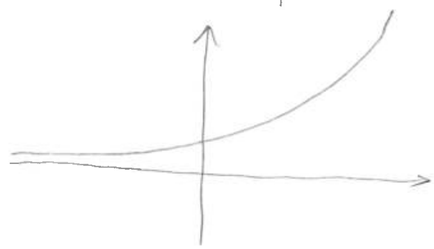
Lecture #9

§ 3.2

Recall that in Week 1, we already introduced $e^z : \mathbb{C} \to \mathbb{C}$ via

$$\boxed{e^{x+iy} = e^x(\cos y + i \cdot \sin y)}$$

Know that e^z - entire function and $\frac{d}{dz} e^z = e^z$. But in contrast to usual calculus where $\exp : \mathbb{R} \to \mathbb{R}$ has a graph



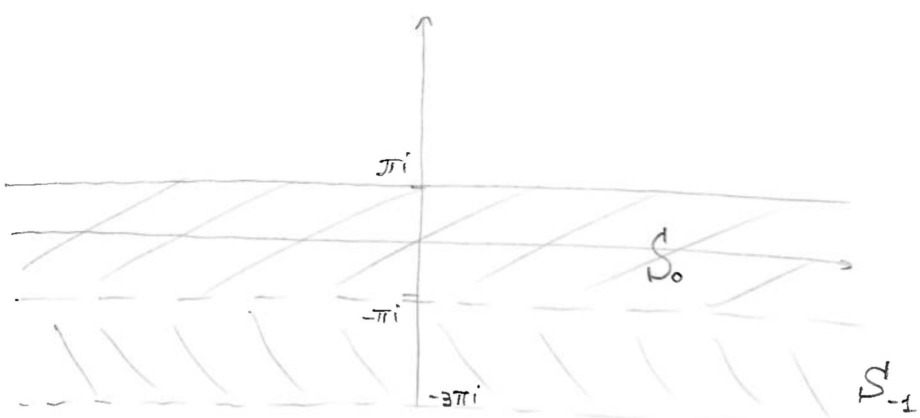
and is clearly one-to-one, this is no longer true now.

$$\boxed{\text{Lemma: } e^{z_1} = e^{z_2} \iff z_1 = z_2 + 2\pi i k \text{ with } k \in \mathbb{Z}}$$

▶ $e^{z_1} = e^{z_2} \iff e^{z_1 - z_2} = 1$.

Let $z_1 - z_2 = x + iy$. Then $e^x \cdot e^{iy} = 1 \Rightarrow x = 0,\ y = 2\pi k$ with $k \in \mathbb{Z}$ ∎

Thus, the complex exponent is a periodic function with period $2\pi i$



Let $S_n = \{x + iy \mid x \in \mathbb{R},\ (2n-1)\pi < y \leq (2n+1)\pi\}$.

Then: 1) $e^z : S_n \to \mathbb{C}$ is one-to-one

2) Range of e^z on S_n is $\mathbb{C} \setminus \{0\}$

For this reason S_n is called "fundamental region" of the exponent map.

Lecture #9

Recall that we already related exponent to trigonometric functions

$$\left.\begin{aligned} e^{i\varphi} &= \cos\varphi + i\sin\varphi \\ e^{-i\varphi} &= \cos\varphi - i\sin\varphi \end{aligned}\right\} \underset{\varphi\in\mathbb{R}}{\Rightarrow} \boxed{\cos\varphi = \frac{e^{i\varphi}+e^{-i\varphi}}{2} \ \& \ \sin\varphi = \frac{e^{i\varphi}-e^{-i\varphi}}{2i}}$$

However, we can now extend cos, sin to entire functions!

Defn: For any $z\in\mathbb{C}$, we define $\cos z = \dfrac{e^{iz}+e^{-iz}}{2}$, $\sin z = \dfrac{e^{iz}-e^{-iz}}{2i}$

True/False: $\cos^2+\sin^2=1$.

▶ True: $\cos^2 z + \sin^2 z = \dfrac{e^{2iz}+e^{-2iz}+2}{4} - \dfrac{e^{2iz}+e^{-2iz}-2}{4} = 1$ ∎

True/False: $|\cos z|$ & $|\sin z| \le 1$

▶ False: e.g. take $z=-10i \Rightarrow \cos(-10i) = \dfrac{e^{10}+e^{-10}}{2} \gg 1$

$$|\sin(-10i)| = \left|\frac{e^{10}-e^{-10}}{2i}\right| \gg 1$$ ∎

Finally: can also define all other trigonometric f-s for $z\in\mathbb{C}$:

$$\boxed{\tan(z) = \frac{\sin(z)}{\cos(z)},\quad \cot(z) = \frac{\cos(z)}{\sin(z)},\quad \sec(z) = \frac{1}{\cos(z)},\quad \csc(z) = \frac{1}{\sin(z)}}$$