Lecture #10

- Comment on homework problem (Ex 6 on p. 77) which states CR in polar coordinates

Approach 1 (suggested by the Hint in the book): Apply the same reasoning as in class, but now using polar coordinates. In other words, given $z_0 = r_0 e^{i\theta_0}$ consider either $z = r_0 e^{i\theta} \to z_0$ with $\theta \to \theta_0$

or $z = r e^{i\theta_0} \to z_0$ with $r \to r_0$.

Approach 2: Derive from the usual CR equations through chain rules, i.e. as $u = u(x,y) = u(r\cos\theta, r\sin\theta) \Rightarrow u_r = u_x \cdot \cos\theta + u_y \cdot \sin\theta$

$$u_\theta = u_x \cdot (-r\sin\theta) + u_y \cdot (r\cos\theta)$$

Ex1 (True/False): e^z is one-to-one on any open disk of radius π.

► True: $e^{z_1} = e^{z_2} \Leftrightarrow z_1 - z_2 = 2\pi i k$ (k-integer)
But z_1, z_2 being inside an open disk of radius $\pi \Rightarrow |z_1 - z_2| < 2\pi$ $\Big\}\Rightarrow z_1 = z_2$ ☒

Ex2: Describe images of horizontal lines (y-fixed) under exponential map.

► Rays! E.g. $z \in \mathbb{R}$ is mapped to $\mathbb{R}_{>0}$,
$z \in \mathbb{R} + i\pi$ is mapped to $\mathbb{R}_{<0}$
$z \in \mathbb{R} + iy$ $(0 < y < \pi)$ is mapped to a ray in upper half-plane
$z \in \mathbb{R} + iy$ $(-\pi < y < 0)$ is mapped to a ray in the lower half-plane ☒

Ex3: Show that if $E'(z) = E(z)$ & $E(0) = 1$, then $E(z) = e^z$
assuming $E(z)$ is an entire f-n.

Note: This shows that the complex exponent can again be defined as a unique solution of diff. equation with initial condition

► • Write $E(z = x+iy) = u(x,y) + i \cdot v(x,y)$. Know (see proof of CR):

$$E'(z) = u_x(x,y) + i v_x(x,y) = v_y(x,y) - i u_y(x,y)$$
$$\parallel$$
$$E(z) = u(x,y) + i \cdot v(x,y)$$

So: $\begin{cases} u_x = u \\ v_x = v \end{cases}$ & $\begin{cases} v_y = u \\ u_y = -v \end{cases}$

↓

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(Continuation)

- Solving $\begin{cases} u_x = u \\ v_x = v \end{cases}$ we get $\begin{cases} u(x,y) = e^x \cdot A(y) \\ v(x,y) = e^x \cdot B(y) \end{cases}$ for some $A, B : \mathbb{R} \to \mathbb{R}$.

- Solving $\begin{cases} v_y = u \\ u_y = -v \end{cases}$ with above data $\Rightarrow \begin{cases} B'(y) = A(y) \\ A'(y) = -B(y) \end{cases}$

- Note that $A''(y) = -B'(y) = -A(y) \Rightarrow A'' + A = 0$. Likewise $B'' + B = 0$.

From differential equations, know that the general solution of $\frac{d^2}{dy^2}\phi(y) + \phi(y) = 0$ is $\phi(y) = a \cdot \cos y + b \cdot \sin y$ with $a, b \in \mathbb{R}$.

So: $B(y) = a\cos y + b\sin y$ for some a, b

$\Downarrow$

$A(y) = B'(y) = b\cos y - a\sin y$

Combining all the above, get:

$$\boxed{E(x+iy) = e^x \cdot ((b\cos y - a\sin y) + i(a\cos y + b\sin y))}$$

- Finally evoking initial condition $E(0) = 1$, we find $b + ia = 1 \Rightarrow \begin{cases} a = 0 \\ b = 1 \end{cases}$, so that

$$E(x+iy) = e^x(\cos y + i\sin y) = e^{x+iy}$$

$\blacksquare$

Bonus Problem: Derive $e^{z+w} = e^z \cdot e^w$ solely by viewing e^z as $E(z)$ above.

Last time: Extended basic trigonometric functions to entire $\mathbb{C}$ via

$$\boxed{\cos(z) = \frac{e^{iz} + e^{-iz}}{2}, \quad \sin(z) = \frac{e^{iz} - e^{-iz}}{2i}}$$

Ex 4: Find zeros of $\cos z$ & $\sin z$.

$\sin z = 0 \Leftrightarrow e^{iz} = e^{-iz} \Leftrightarrow e^{2iz} = 1 \overset{\text{last time}}{\Longleftrightarrow} 2iz = 2i\pi k \ (k \in \mathbb{Z}) \Leftrightarrow z = \pi k$ (k-integer)

$\cos z = 0 \Leftrightarrow e^{iz} = -e^{-iz} \Leftrightarrow e^{2iz} = -1 = e^{\pi i} \Leftrightarrow 2iz = 2i\pi k + i\pi \ (k \in \mathbb{Z}) \Leftrightarrow z = \pi k + \frac{\pi}{2}$ (k integer)

$\blacksquare$

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With this in mind, we can also extend other trig. functions to $\mathbb{C}$:

$\tan(z) = \frac{\sin z}{\cos z}$, $\sec(z) = \frac{1}{\cos z}$ — analytic in $\mathbb{C} \setminus \{\pi k + \frac{\pi}{2} \mid k\text{-integer}\}$

$\cot(z) = \frac{\cos z}{\sin z}$, $\csc(z) = \frac{1}{\sin z}$ — analytic in $\mathbb{C} \setminus \{\pi k \mid k\text{-integer}\}$

Ex 5: Verify that derivatives of these f-s are given by usual formulas

The trig. functions also admit so-called "hyperbolic" analogues, many of you already saw them before:

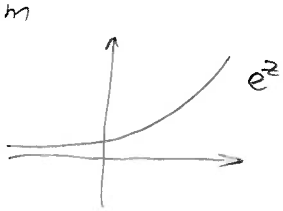
$$\boxed{\begin{array}{ll} \text{hyperbolic sine} & \sinh(z) = \frac{e^z - e^{-z}}{2} \\ \text{hyperbolic cosine} & \cosh(z) = \frac{e^z + e^{-z}}{2} \\ \text{hyperbolic tangent} & \tanh(z) = \frac{\sinh z}{\cosh z} = \frac{e^z - e^{-z}}{e^z + e^{-z}} \end{array}}$$

Key Point: While in the usual calculus you have the whole "zoo" of trigonometric and hyperbolic trig. f-s, when viewing them over $\mathbb{C}$, we actually see all of them are expressed via complex exponent.

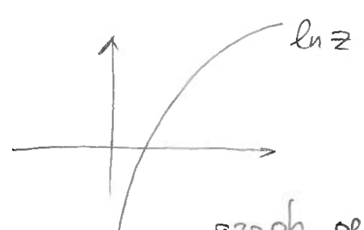
Note: $$\boxed{\begin{array}{l} \sinh(iz) = i \cdot \sin(z) \ \ \& \ \ \cosh(iz) = \cos z \\ \sin(iz) = i \cdot \sinh(z) \ \ \& \ \ \cos(iz) = \cosh(z) \end{array}}$$

§3.3 Logarithm

In school:



graph of $e^z : \mathbb{R} \to \mathbb{R}$



graph of $\ln z : \mathbb{R}_{>0} \to \mathbb{R}$

Usually $\ln z$ is defined as inverse of e^z (well-defined as e^z is 1-to-1)

Now: $e^z : \mathbb{C} \to \mathbb{C}$ is not 1-to-1, so the inverse is not a well-defined function but is rather a multivalued function.

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Let's solve $z=e^w$ with z given to us. If $w=x+iy$ while $z=re^{i\theta}$, then

$$\begin{cases} r=e^x \\ e^{i\theta}=e^{iy} \end{cases} \iff \begin{cases} x=\ln r \\ y=\theta+2\pi k,\ k\text{-integer} \end{cases}$$

Following the book notations, we shall use $\mathrm{Log}\, r := \ln r$ when $r\in\mathbb{R}_{>0}$.

Def: For any $z\in\mathbb{C}\setminus\{0\}$, $\log z$ is actually a set of all w as above:

$$\boxed{\log z = \mathrm{Log}|z| + i\cdot \arg(z) = \{\mathrm{Log}|z| + i\cdot \mathrm{Arg}(z) + i\cdot 2\pi k\ (k\text{-integer})\}}$$

Note: 1) $\log(e^z)\neq z$ but rather $\log(e^z)=\{z+2\pi i\cdot k\ (k\text{-integer})\}$

2) $\log(z_1z_2)=\log(z_1)+\log(z_2)$, $\log\left(\frac{z_1}{z_2}\right)=\log(z_1)-\log(z_2)$

which follows from similar properties of $\arg(z)$

Recall: We already saw such a phenomena back in Lecture 1, i.e. $\arg(z)$ is a multi-valued function, while we used principal argument $\mathrm{Arg}(z)$ for specific choice ($\in(-\pi,\pi]$).

Def: The principal value of the logarithm is

$$\boxed{\mathrm{Log}\, z = \mathrm{Log}|z| + i\,\mathrm{Arg}(z)}$$

note it's continuous only on $\mathbb{C}\setminus\mathbb{R}_{\le 0}$.

But one could completely analogously choose any other branch of $\arg(z)$ which should also give rise to a branch of $\arg(z)$. Namely:

- for any angle τ, define $\arg_\tau(z)$ to be the unique value of $\arg(z)$ that lies in $(\tau,\tau+2\pi]$, e.g. $\arg_{-\pi}(z)=\mathrm{Arg}(z)$. Then $\arg_\tau(z)$ is continuous on $\mathbb{C}\setminus\{\text{ray } \theta=\tau\}$.
- given such τ, we also can define

$$\mathrm{Log}_\tau z = \mathcal{L}_\tau(z) = \mathrm{Log}|z| + i\arg_\tau(z).$$