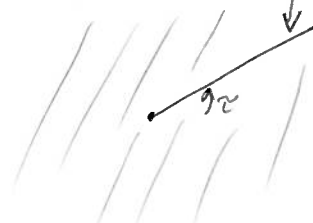
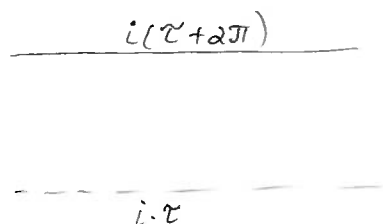
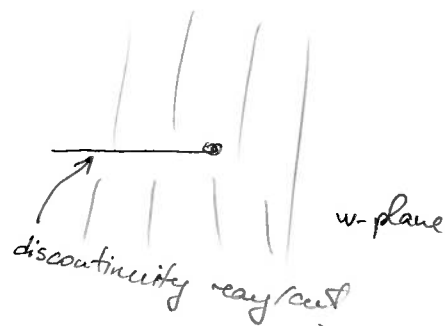
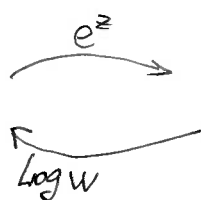
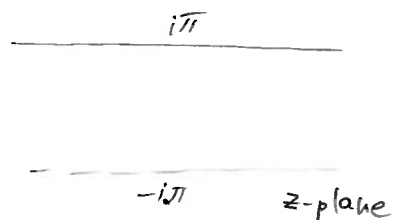


Lecture #11 (§3.3-3.4)

1

Last time:



Theorem 1: $\text{Log } z$ is analytic in $D^* = \mathbb{C} \setminus \mathbb{R}_{\leq 0}$

Recall from Homework that Cauchy-Riemann in polar coordinates is:

$$\begin{cases} u_r = \frac{1}{r} v_\theta \\ v_r = -\frac{1}{r} u_\theta \end{cases}$$

where $u(r, \theta) + i v(r, \theta) = f(re^{i\theta})$

$\underbrace{f}$ $\mathbb{C}$ -valued function whose analyticity we check.

In our case: $\text{Log } z = \text{Log } |z| + i \cdot \text{Arg } z$

$$\begin{cases} u(r, \theta) = \text{Log } r = \ln r \\ v(r, \theta) = \theta \text{ with } \theta \in (-\pi, \pi] \end{cases}$$

$$\Rightarrow \begin{cases} u_r = \frac{1}{r} = \frac{1}{r} v_\theta \\ v_r = 0 = -\frac{1}{r} u_\theta \end{cases} \quad \checkmark$$

Note: The same clearly holds for $\text{Log}_\tau z$ with D^* being rather replaced by $D_\tau^* = \mathbb{C} \setminus \{ \text{ray } \theta = \tau \}$

Theorem 2: $\frac{d}{dz} \text{Log } z = \frac{1}{z}$ in D^* (and similarly $\frac{d}{dz} \text{Log}_\tau z = \frac{1}{z}$ in D_τ^*)

Apply $\frac{d}{dz}$ to the equality $z = e^{\text{Log } z}$ to get $1 = e^{\text{Log } z} \cdot \frac{d}{dz} (\text{Log } z) = z \cdot (\text{Log } z)'$

$$\Rightarrow (\text{Log } z)' = \frac{1}{z}$$

Rmk: Using chain rule we see that $\begin{pmatrix} u_r & \frac{1}{r} u_\theta \\ v_r & \frac{1}{r} v_\theta \end{pmatrix} = \begin{pmatrix} u_x & u_y \\ v_x & v_y \end{pmatrix} \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix}$

polar $\underline{CR}$: $\begin{pmatrix} A & -B \\ B & A \end{pmatrix}$ $\underline{CR}$: $\begin{pmatrix} a & -b \\ b & a \end{pmatrix}$

Lecture #11

12

Ex 1: Determine domain of analyticity of $f(z) = \log(5z - 2i)$. Find $f'(z)$

$$\triangleright 5z - 2i \notin \mathbb{R}_{\leq 0} \Leftrightarrow z \notin \mathbb{R}_{\leq 0} + \frac{2}{5}i$$

$$f'(z) = \frac{1}{5z - 2i} \cdot 5 = \frac{1}{z - \frac{2i}{5}}$$

Ex 2: a) Show that $\log \sqrt{x^2 + y^2}$ is a harmonic function on $\mathbb{R}^2 \setminus \{0\}$

b) Verify that $\text{Arg } z$ is a its harmonic conjugate on $D^* = \mathbb{R}^2 \setminus \{(x, 0) \mid x \in \mathbb{R}_{\leq 0}\}$

c) Show that there is no harmonic conjugate on all $\mathbb{R}^2 \setminus \{0\}$.

$$\triangleright a) \log \sqrt{x^2 + y^2} = \text{Re}(\log z) \Rightarrow \text{harmonic on } D$$

$$\text{Likewise } \log \sqrt{x^2 + y^2} = \text{Re}(\log_0 z) \Rightarrow \text{harmonic on } \mathbb{R}^2 \setminus \{(x, 0) \mid x \in \mathbb{R}_{\leq 0}\} \Rightarrow \text{it's harmonic on all } \mathbb{R}^2 \setminus \{(0, 0)\}.$$

$$b) \text{Arg } z = \text{Im}(\log z) \Rightarrow \text{any harmonic conjugate on } D^* \text{ is } \text{Arg } z + C \quad \left\{ \begin{array}{l} \text{constant} \end{array} \right. \Rightarrow$$

c) It would be one of those in b), but $\text{Arg } z$ is discontinuous on the ray $\mathbb{R}_{\leq 0} \Rightarrow$ no harmonic conjugate in $\mathbb{R}^2 \setminus \mathbb{R}_{\leq 0}$.

Ex 3: Express inverse to cosine via logarithm.

$$\triangleright \text{If } z = \cos^{-1}(w), \text{ then it's equivalent to } w = \cos z = \frac{e^{iz} + e^{-iz}}{2}. \text{ The latter is equivalent to } e^{2iz} - 2w \cdot e^{iz} + 1 = 0 \Rightarrow e^{iz} = \frac{2w \pm \sqrt{4w^2 - 4}}{2} = w \pm \sqrt{w^2 - 1} \quad \left\{ \begin{array}{l} \Rightarrow \end{array} \right.$$

$$\text{But: } \sqrt{\dots} \text{ is already a multivalued function } \Rightarrow \text{no need for } \pm$$
$$\Rightarrow e^{iz} = w + \sqrt{w^2 - 1} \Leftrightarrow iz = \log(w + \sqrt{w^2 - 1}) \Leftrightarrow \boxed{z = -i \log(w + \sqrt{w^2 - 1})}$$

composition of multivalued functions

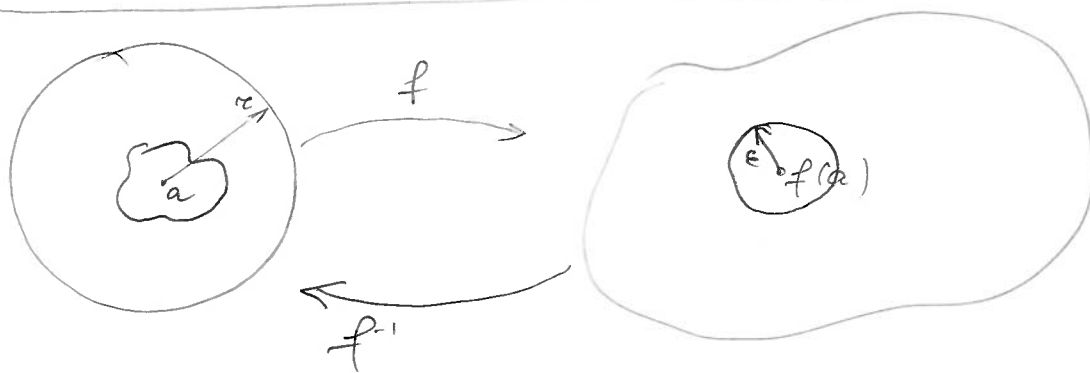
$$\left[\text{Note: } \frac{d}{dw} \cos^{-1} w = -i \cdot \frac{1}{w + \sqrt{w^2 - 1}} \left(1 + \frac{w}{\sqrt{w^2 - 1}} \right) = \frac{-i}{\sqrt{w^2 - 1}} = -\frac{1}{\sqrt{1 - w^2}} \leftarrow \text{as in school!} \right.$$

Lecture #11

13

In fact our proofs of Thms 1 & 2 can be naturally generalized:

Theorem 3 ("Complex inverse function Theorem"): If f is analytic in $D_r(a)$ and $f'(a) \neq 0$, then f^{-1} is well-defined in a small neighborhood $D_\epsilon(f(a))$ and f^{-1} is analytic there with $\frac{d}{dw} f^{-1}(w) = \frac{1}{f'(f^{-1}(w))}$



1) By the inverse function theorem in real analysis it suffices to check

$$\det \begin{pmatrix} u_x & u_y \\ v_x & v_y \end{pmatrix} \neq 0 \quad \text{where} \quad f(x+iy) = u(x,y) + i v(x,y).$$

$$\text{But CR} \Rightarrow u_x = v_y, \quad v_x = -u_y \Rightarrow \det \begin{pmatrix} u_x & u_y \\ v_x & v_y \end{pmatrix} = (u_x)^2 + (v_x)^2 = |u_x + i v_x|^2 = |f'(a)|^2.$$

The result now follows as $f'(a) \neq 0$

2) Follows from chain rule by applying $\frac{d}{dw}$ to $w = f(f^{-1}(w))$

§3.4 Washers, wedges, walls

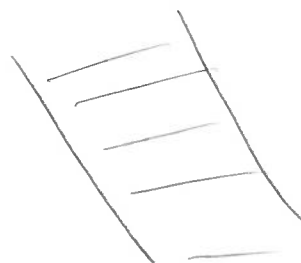
As we shall see much later in the course, often solving Laplace's equation $(\frac{d^2}{dx^2} + \frac{d^2}{dy^2}) \phi(x,y) = 0$ can be reduced to the domain:



"washer"



"wedge"



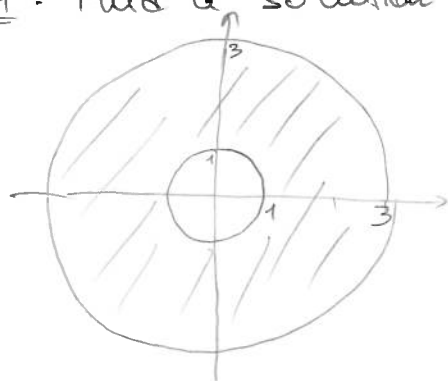
"walls"

Lecture #11

4

Let's show how one can find some of such solutions with boundary conditions.

Ex 4: Find a solution $\phi(x,y)$ of Laplace equation in the domain with initial condition:



$$\phi|_{\text{inner circle}} = 10$$

$$\phi|_{\text{outer circle}} = 40$$

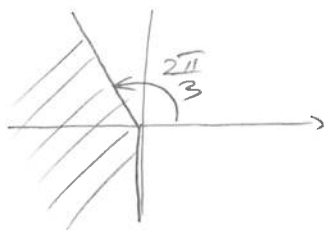
As ϕ is constant on these two circles it's logical to search for ϕ as a function of $\sqrt{x^2+y^2} = r$. Best by Ex 2(a) know $\log \sqrt{x^2+y^2}$ is harmonic.

Idea: For any constants A, B a function $A \cdot \log \sqrt{x^2+y^2} + B$ is also harmonic!

Initial conditions:
$$\begin{cases} A \cdot \log 1 + B = 10 \Rightarrow B = 10 \\ A \cdot \log 3 + B = 40 \end{cases} \Rightarrow A = \frac{30}{\ln 3}$$

So: $\phi(x,y) = \frac{30}{\ln 3} \cdot \log \sqrt{x^2+y^2} + 10$ is a solution

Ex 5: Find a solution $\phi(x,y)$ of Laplace's equation in the domain with initial conditions



$$\phi|_{\text{ray with angle } \frac{2\pi}{3}} = 10$$

$$\phi|_{\text{ray with angle } -\frac{\pi}{2}} = 60$$

Idea: These two rays are level curves of function θ , hence it's logical to look for the answer in the form $A \cdot \text{Arg } z + B$.

Warning: $\text{Arg } z$ is discontinuous on $\mathbb{R}_{\leq 0}$ inside our region though!

► A simple fix to the above issue is to take a different branch!

E.g. consider $A \cdot \arg_0 z + B$

harmonic in $\mathbb{C} \setminus \mathbb{R}_{\geq 0}$

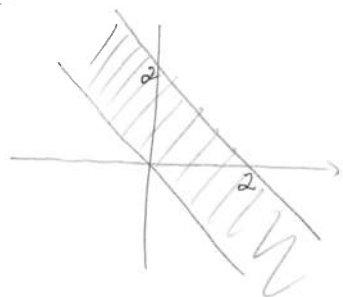
Find A, B from initial condition:

$$\begin{cases} A \cdot \frac{2\pi}{3} + B = 10 \\ A \cdot \frac{3\pi}{2} + B = 60 \end{cases} \xRightarrow{\text{solve}} \begin{cases} A = \frac{60}{\pi} \\ B = -30 \end{cases}$$

↑ Not $-\frac{\pi}{2}$ bc of our branch!

So: $\boxed{\frac{60}{\pi} \arg_0 z - 30}$ is a solution!

Ex 6: Find a solution of Laplace eqn in the depicted region with initial condition



$$\phi|_{x+y=0} = 10$$

$$\phi|_{x+y=2} = 20$$

► Clearly above lines are level curves of $x+y$. Any linear function is harmonic clearly! Hence, let's look for $A(x+y) + B$.

Initial condition: $\begin{cases} A \cdot 0 + B = 10 \\ A \cdot 2 + B = 20 \end{cases} \Rightarrow \begin{cases} A = 5 \\ B = 10 \end{cases}$

So: $\phi(x,y) = 5(x+y) + 10$ is a solution!

Rmk: For washers and wedges centered at some $a \neq 0$ use the same idea by replacing $\log|z| \rightsquigarrow \log|z-a|$
 $\arg z \rightsquigarrow \arg(z-a)$
 relevant for homework.