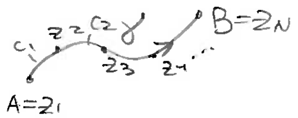
Lecture #13

Similarly to calculus one defines integrals formally as

$$\int_\gamma f(z)dz = \lim_{\substack{\text{max length} \\ \text{b/w } z_i \,\&\, z_{i+1} \to 0}} \left\{ \sum_{i=1}^{N-1} f(c_i)\cdot(z_{i+1}-z_i) \right\}$$



While it's quite useless in practical computations, let's note some properties.

$$\boxed{\begin{array}{l} \int_\gamma c\cdot f(z)dz = c\cdot\int_\gamma f(z)dz, \quad \int_\gamma (f(z)+g(z))dz = \int_\gamma f(z)dz + \int_\gamma g(z)dz \\ \int_{-\gamma} f(z)dz = -\int_\gamma f(z)dz \quad \text{with } -\gamma \text{ being oppositely oriented path} \end{array}}$$

For practical reasons, we shall actually take another viewpoint. Note that $z_{i+1}-z_i \sim z'(t_i)\cdot(t_{i+1}-t_i)$ where $z:[a,b]\longrightarrow\gamma$ - parametrization and $z_i = z(t_i)$

Therefore, for us:

$$\boxed{\int_\gamma f(z)dz = \int_a^b f(z(t))\cdot z'(t)dt} \leftarrow \text{where } z:[a,b]\longrightarrow\gamma \text{ is some parametrization of smooth curve } \gamma$$

Note: In particular, the result is actually independent of parametriz!

If Γ is a oriented contour, i.e. $\Gamma = \gamma_1+\gamma_2+\_\_+\gamma_n$ with γ_i - smooth curves, then from above limit abstract definition it's clear that

$$\boxed{\int_\Gamma f(z)dz = \int_{\gamma_1} f(z)dz + \_\_ + \int_{\gamma_n} f(z)dz}$$

In particular, if $\gamma = [a,b]\subseteq\mathbb{R}\subseteq\mathbb{C}$ then choosing $z:[a,b]\to\gamma$, $t\mapsto t$ we get $\int_{[a,b]} f(z)dz = \int_a^b f(t)dt$ which is just $\int_a^b u(t,0)dt + i\cdot\int_a^b v(t,0)dt$ if $f(x+iy) = u(x,y)+i\cdot v(x,y)$

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Ex 1: a) Evaluate $\int_{\gamma'} e^{3z}dz$ with γ' being line segment from 1 to -1

b) Evaluate $\int_{\gamma} e^{3z}dz$ with γ being an upper arch of unit circle from 1 to -1.

a) Know that $e^{3x} = \left(\frac{1}{3}e^{3x}\right)'$, hence by Fundamental Theorem of Calculus:

$$\int_{\gamma'} e^{3z}dz = \left(\frac{1}{3}e^{3x}\right)\Big|_{x=1}^{x=-1} = \frac{e^{-3}-e^{3}}{3}$$

b) If we were to use direct definition, then first we parametrize γ

E.g. $z: [0,\pi] \to \gamma$, $t \mapsto \cos t + i\sin t$. Then, our integral is just

$$\int_0^{\pi} e^{3(\cos t + i\sin t)} \cdot (-\sin t + i\cos t)dt = \int_0^{\pi} e^{3\cos t}\left(\cos(\sin t) + i\cdot\sin(\sin t)\right) \cdot (-\sin t + i\cos t)dt = \ldots ?$$

But good news we can still apply Fund. Thm. Calculus, since

$$e^{3z} = \left(\frac{1}{3}e^{3z}\right)' \text{ for all } z!$$

$$\Rightarrow \int_{\gamma} e^{3z}dz = \frac{e^{3z}}{3}\Big|_{z=1}^{z=-1} = \frac{e^{-3}-e^{3}}{3}$$

Fundamental Theorem of Calculus in present setup = [§ 4.3, Thm 6]:

If $f(z)$ is a $\mathbb{C}$-valued continuous function in a domain D and has antiderivative $F(z)$ on D, i.e. $F'(z) = f(z)$, then for any contour $\Gamma = \Gamma_A^B$ (A = starting point, B = end-point) lying in D, we have

$$\int_{\Gamma} f(z)dz = F(B) - F(A)$$

Corollaries: 1) $\int_{\Gamma} f(z)dz = 0$ for closed Γ in above setup

2) $\int_{\Gamma_A^B} f(z)dz$ depends only on A, B not a curve, hence, we can replace Γ accordingly (like γ' instead of γ in Ex 1!)

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Proof of Thm

1) Assume first γ - smooth curve with parametrization $z:[a,b]\to\gamma$
$z(t)=x(t)+i\,y(t)$

Let's evaluate $\frac{d}{dt}(F(z(t)))$, where $F(x+iy)=u(x,y)+i\,v(x,y)$.

$$\frac{d}{dt}(F(z(t)))=\frac{d}{dt}\left(u(x(t),y(t))+i\cdot v(x(t),y(t))\right) \stackrel{\text{usual chain rule}}{=}$$

$$=u_x\cdot x'+u_y\cdot y'+i\,(v_x\cdot x'+v_y\cdot y') \stackrel{CR}{=}$$

$$=u_x\cdot x'-v_x\cdot y'+i\,(v_x\cdot x'+u_x\cdot y')=\underbrace{(u_x+i\,v_x)}_{F'}(x'+iy')$$

$$\Downarrow$$

$$\frac{d}{dt}F(z(t))=F'(z(t))\cdot z'(t)$$

So: $$\int_{\gamma_A^B} f(z)dz=\int_a^b \underbrace{f(z(t))}_{=F'(z(t))}\cdot z'(t)dt=\int_a^b\frac{d}{dt}F(z(t))\,dt \stackrel{\text{usual Fund. Thm. Calculus}}{=} F(\underbrace{z(b)}_{=B})-F(\underbrace{z(a)}_{=A})$$

$$\Downarrow$$

$$\boxed{\int_{\gamma_A^B} f(z)dz=F(B)-F(A)}$$

2) Now if $\Gamma=\gamma_1+\gamma_2+\ldots+\gamma_n$, with γ_i from z_i to z_{i+1}, then:

$$\int_\Gamma f(z)dz=\int_{\gamma_1}f(z)dz+\ldots+\int_{\gamma_n}f(z)dz=F(z_2)-F(z_1)+F(z_3)-F(z_2)+\ldots+F(z_{n+1})-F(z_n)$$

$$\stackrel{\text{telescopic sum}}{=} F(\underset{B}{\underset{\shortparallel}{z_{n+1}}})-F(\underset{A}{\underset{\shortparallel}{z_1}})=F(B)-F(A)$$

Note: The above result applies only when antiderivative exists!
If not, then just do direct calculation.

Fact: If $f(z)$ is continuous on γ, then $\int_\gamma f(z)dz$ exists

(can you deduce it from a similar result in calculus?)

Lecture #13

The following is a mandatory exercise on the subject:

Ex2: Let $C_r(z_0)$ be a circle centered at z_0 of radius r oriented counter clockwise, and n be an integer. Compute

$$\int_{C_r(z_0)} (z-z_0)^n dz.$$

Proof 1

First parametrize $C_r(z_0)$ via e.g. $z(t) = z_0 + re^{it}$ with $0 \le t \le 2\pi$

Then $(z-z_0)^n = (re^{it})^n = r^n \cdot e^{int}$, $z'(t) = r \cdot i \cdot e^{it}$

$$\Rightarrow \int_0^{2\pi} r^n \cdot e^{int} \cdot ri \cdot e^{it} dt = r^{n+1} \cdot i \int_0^{2\pi} e^{i(n+1)t} dt$$

Case 1: $n \neq -1 \Rightarrow$ antiderivative of $e^{i(n+1)t}$ is $\frac{e^{i(n+1)t}}{i(n+1)}$

$$\Rightarrow \int_0^{2\pi} e^{i(n+1)t} dt = \frac{e^{i(n+1)t}}{i(n+1)} \Big|_{t=0}^{t=2\pi} = 0$$

Case 2: $n = -1 \Rightarrow$ get $i \int_0^{2\pi} 1 dt = 2\pi i$

Answer: $\int_{C_r(z_0)} (z-z_0)^n dz = \begin{cases} 0, & n \neq -1 \\ 2\pi i, & n = -1 \end{cases}$

Proof 2

- If $n \neq -1$, then $(z-z_0)^n = \left(\frac{1}{n+1}(z-z_0)^{n+1}\right)'$ which holds on all $\mathbb{C}$ with a possible exception of z_0 when $n<0$

Hence by Fund. Thm. Calculus (see Corollary 1)): $\int_{C_r(z_0)} (z-z_0)^n dz = 0$.

- If $n=-1$, then $(z-z_0)^{-1} = \frac{1}{z-z_0}$ does <u>not</u> have an antiderivative on any domain containing $C_r(z_0)$ since $\log(z-z_0)$ has no branch on $\mathbb{C} \setminus \{z_0\}$. However, if we split $C_r(z_0)$ into 2 archs you can still apply Fundam. Thm. Calculus – finish at home!