Lecture #14

Ex 1: Compute $\int_\Gamma f(z)dz$ where $f(z)=\frac{1}{z}$ and contour is:

a) Γ is a half-circle from 1 to -1

b) Γ is a unit circle oriented counterclockwise

c) Γ is an ellipse $4x^2+y^2=1$ oriented counterclockwise.

a) Proof 1 (as last time): parametrize $z:[0,\pi]\to\Gamma$, $t\longmapsto \cos t+i\sin t=e^{it}$

Then $\int_\Gamma \frac{1}{z}dz=\int_0^\pi e^{-it}\cdot ie^{it}dt=i\pi$

Proof 2 (using Fundamental Theorem of Calculus = "F.T.C.")

$Log_{-\frac{\pi}{2}}z$ is an antiderivative of $\frac{1}{z}$ on $\mathbb{C}\setminus\{\text{ray }\theta=-\pi/2\}$ which contains Γ

$\Rightarrow \int_\Gamma \frac{1}{z}dz = Log_{-\pi/2}z\Big|_{z=1}^{z=-1}=(\ln 1+i\arg_{-\pi/2}(-1))-(\ln 1+i\arg_{-\pi/2}1)=\pi i$

b) We did the computation in a direct way last time to get $2\pi i$.

Also we noted the argument of Proof 2 from a) does not apply on the nose, as $\log z$ has no branch on open set containing $\underbrace{|z|=1}_{\text{unit circle}}$.

Fix 1: You can split the circle into two arches (e.g. top & bottom):

$$\int_{\circ}\frac{1}{z}dz=\underbrace{\int_{\frown}\frac{1}{z}dz}_{=\pi i\text{ by a)}}+\underbrace{\int_{\smile}\frac{1}{z}dz}_{\substack{\text{can be computed as in a) but we need a branch of}\\ \text{log that contains } \smile,\text{ e.g. } Log_{\frac{\pi}{2}}z.}} = \pi i+\pi i=2\pi i$$

$\Rightarrow \int_{\smile}\frac{1}{z}dz=Log_{\frac{\pi}{2}}z\Big|_{-1}^{1}=i\cdot 2\pi-i\cdot\pi=\pi i$

B & A approach $e^{i\cdot(-\pi/2)}$

Fix 2: $\int_{\circ}\frac{1}{z}dz=\lim\int\frac{1}{z}dz=\lim\left(Log_{-\frac{\pi}{2}}z\Big|_A^B\right)=i\cdot\frac{3\pi}{2}-i\cdot\left(-\frac{\pi}{2}\right)=2\pi i.$

c) Proof 1 fails, but either of Fix 1 or Fix 2 apply to give the same $2\pi i$

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As we shall see next week the above $\int_\Gamma \frac{1}{z}dz$ is key to all course!

Next week, in the proof of several important results we would need bounds on integrals. To this end, recall that in calculus

$$\left|\int_a^b f(z)dz\right| \leq M\cdot |b-a| \quad \text{if} \quad |f(z)|\leq M \text{ for all } a\leq z\leq b.$$

$$\boxed{\text{Lemma: } \left|\int_\Gamma f(z)dz\right| \leq \max_{z\in\Gamma}|f(z)|\cdot \text{length}(\Gamma)}$$

▸ This follows immediately from above calculus result:

$$\left|\int_\Gamma f(z)dz\right| \leq \int_a^b |f(z)|\cdot|z'(t)|dt \leq \max_{z\in\Gamma}|f(z)|\cdot \underbrace{\int_a^b \underbrace{\sqrt{x'(t)^2+y'(t)^2}\,dt}_{ds}}_{\text{length}(\Gamma)}$$

parametrize $z:[a,b]\to\Gamma$
$z(t)=x(t)+i\cdot y(t)$

∎

Ex2: Show that $\left|\int_C \frac{dz}{z^2-i}\right| \leq \frac{3\pi}{4}$ where C is a circle $|z|=3$ oriented counterclockwise

▸ $\text{length}(C)=6\pi$.

As $|z|=3 \Rightarrow |z^2|=9 \Rightarrow z^2-i$ is a circle of radius 9 centered at $-i$

$\Rightarrow |z^2-i|\geq 8$ (from triangle inequality or just draw a picture)

$\Rightarrow \frac{1}{|z^2-i|}\leq\frac{1}{8}$.

Hence by Lemma, we get $\left|\int_C \frac{dz}{z^2-i}\right| \leq 6\pi\cdot\frac{1}{8} = \frac{3\pi}{4}$

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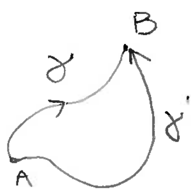
Theorem 1: Let f be a continuous function on a domain D. The following are equivalent:

1) f has an antiderivative in D
2) $\int_{\gamma_A^A} f(z)dz = 0$ for any closed contour $\gamma = \gamma_A^A$
3) $\int_{\gamma_A^B} f(z)dz = \int_{(\gamma')_A^B} f(z)dz$, i.e. path-independence.

► 1) $\Rightarrow$ 2) by F.T.C.

2) $\Rightarrow$ 3):



equivalent to $\underbrace{\int_{\gamma_A^B} f dz - \int_{(\gamma')_A^B} f dz}_{=} = 0$

$\int f(z)dz = 0$ as $\gamma + (-\gamma')$ - closed.

3) $\Rightarrow$ 1): key is where to search for antiderivative?!

But the guess is dictated by F.T.C: $\int_{\gamma_a^z} f(w)dw = F(z) - F(a)$ if F was an antiderivative of f. We shall use this as definition.

Fix any a in D, and consider $\boxed{F(z) := \int_{\gamma_a^z} f(w)dw}$

- property 3) guarantees the result is independent of γ_a^z
- different choice of a will change F only by adding constant

Remarks: verify that $\boxed{F'(z_0) = f(z_0) \text{ for any } z_0 \in D}$

Recall that $F'(z_0) = \lim_{z \to z_0} \frac{F(z) - F(z_0)}{z - z_0} = \lim_{z \to z_0} \frac{\int_{\gamma_{z_0}^z} f(w)dw}{z - z_0}$

The easiest way is to take γ_a^z as $\gamma_a^{z_0} + \gamma_{z_0}^z$

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(Continuation of the proof)

As f is continuous near $z_0 \Rightarrow f(z) = f(z_0) + g_1(z)$ and $g_1(z) \to 0$ as $z \to z_0$.

Thus: $\int_{\gamma_{z_0}^{z}} f(w)\,dw = f(z_0)\cdot \int_{\gamma_{z_0}^{z}} 1\,dw + \int_{\gamma_{z_0}^{z}} g_1(w)\,dw = (z-z_0)f(z_0) + \int_{\gamma_{z_0}^{z}} g_1(w)\,dw.$

Hence:
$$\frac{\int_{\gamma_{z_0}^{z}} f(w)\,dw}{z-z_0} - f(z_0) = \frac{\int_{\gamma_{z_0}^{z}} g_1(w)\,dw}{z-z_0}.$$

As $g_1(w) \to 0$ as $w \to z_0$ we get $|g_1(w)| < \varepsilon$ (any) for w in small disk around z_0. But choosing $\gamma_{z_0}^{z}$ to be a straight line segment and using Lemma on p. 2, we get $\left|\frac{\int_{\gamma_{z_0}^{z}} g_1(w)\,dw}{z-z_0}\right| < \varepsilon$

This proves $F'(z_0) = f(z_0)$, as claimed ■

While the above result is cool, one may ask how do we check in practice if properties 2) or 3) hold. A partial answer is given by the following Key Result:

> **Theorem 2** ("Cauchy's Theorem"): Suppose $f(z)$ is analytic on a simple closed curve γ as well as inside it. Then $\int_{\gamma} f\,dz = 0$

> **Corollary**: Thm 1 + Thm 2 $\Rightarrow$ analytic functions on interior of simple closed curve always have analytic antiderivative.

! **Warning**: It's key that $f(z)$ is analytic not only on γ but also in interior as example of $\int_{\odot} \frac{1}{z}\,dz \neq 0$ shows

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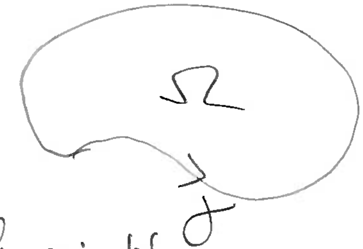
The proof of this theorem is based on the Green's Theorem:

$$\boxed{\int_\gamma P\,dx + Q\,dy = \iint_\Omega (Q_x - P_y)\,dA}$$



here γ is a simple closed curve positively oriented

Ω = interior of γ, $dA = dx\,dy$

and we assume $Q_x = \frac{\partial Q}{\partial x}$, $P_y = \frac{\partial P}{\partial y}$ are continuous on Ω (and its open nbhd).

Recall: If $z : [a,b] \to \gamma$ is a parametrization, then $z(t) = (x(t), y(t))$

$$\int_\gamma P\,dx + Q\,dy = \int_a^b \left[P(x(t), y(t)) \cdot x'(t) + Q(x(t), y(t)) \cdot y'(t)\right] dt$$

Proof of Theorem 2

$f(x+iy) = u(x,y) + i \cdot v(x,y)$

Note: f-analytic $\Rightarrow u, v - C^1$-smooth $\Rightarrow$ Green thm applies.

So: $$\int_\gamma f(z)\,dz = \int_a^b \big(u(x(t), y(t)) + i \cdot v(x(t), y(t))\big) \cdot (x'(t) + i y'(t))\,dt$$

$$= \int_a^b (u \cdot x' - v \cdot y')\,dt + i \int_a^b (v x' + u y')\,dt$$

$$= \int_\gamma (u\,dx - v\,dy) + i \cdot \int_\gamma (v\,dx + u\,dy) \quad \underline{\underline{\text{Green's Theorem}}}$$

$$= \iint_\Omega (-v_x - u_y)\,dA + i \iint_\Omega (u_x - v_y)\,dA$$

But Cauchy-Riemann eqs $\Rightarrow u_x - v_y = 0 = -v_x - u_y$

Hence: $\int_\gamma f\,dz = 0$