

Lecture #15

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• Start with reminder of two key theorems from last time.

• Ex 1: Evaluate $\int_{\Gamma} \frac{dz}{z}$ where Γ is a unit circle $|z|=1$ traversed twice clockwise.

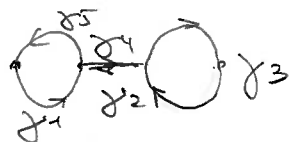
• Know: $\int_{\gamma} \frac{dz}{z} = 2\pi i$ where γ is a unit circle $|z|=1$ traversed once counter clockwise.

$$\Rightarrow \int_{\Gamma} \frac{dz}{z} = -2 \cdot \int_{\gamma} \frac{dz}{z} = -4\pi i$$

• TIF: Can the claim of Thm 2 from last time be generalized by replacing "simple closed curve" with "closed contour".

Answer: Yes, let's illustrate this with the following example.

Let Γ be the curve $\gamma_1 + \gamma_2 + \gamma_3 + \gamma_4 + \gamma_5$



$$\text{Then } \int_{\Gamma} = \int_{\gamma_1} + \int_{\gamma_2} + \int_{\gamma_3} + \int_{\gamma_4} + \int_{\gamma_5} = \underbrace{\left(\int_{\gamma_1} + \int_{\gamma_5} \right)}_{=0 \text{ by Thm 2}} + \underbrace{\left(\int_{\gamma_2} + \int_{\gamma_4} \right)}_{=0} + \underbrace{\int_{\gamma_3}}_{=0 \text{ by Thm 2}}$$

This can be summarized by

Theorem 1: Suppose Ω is a domain without holes and let f be analytic in Ω . Then $\int_{\gamma} f dz = 0$ for any closed contour γ in Ω .

Corollary: Analytic functions in domains without holes have antiderivatives.

Warning: This is no longer true in all domains, as illustrated by $f(z) = \frac{1}{z}$ in the washer $\Omega = \{z \in \mathbb{C} \mid 1 < |z| < 2\}$.

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Ex2: Let Γ be a simple closed contour (on the complex plane) positively oriented and $a \notin \Gamma$. Find $\int_{\Gamma} \frac{1}{z-a} dz$.

Case 1: $a \in \text{outside of } \Gamma$.

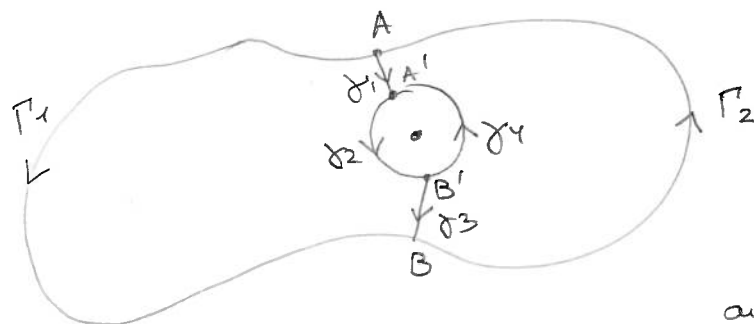
In this case $\frac{1}{z-a}$ is analytic on Γ & inside $\Rightarrow \int_{\Gamma} \frac{1}{z-a} dz = 0$ (Thm 1)

Case 2: $a \in \text{inside of } \Gamma$.

We know that if γ is a circle (of any radius), centered at a and positively oriented then $\int_{\gamma} \frac{1}{z-a} dz = 2\pi i$.

Claim: $\int_{\Gamma} \frac{1}{z-a} dz = \int_{\gamma} \frac{1}{z-a} dz$

Let's illustrate it by picture-type argument:



Pick points A, A', B, B' as on picture. Let γ_1 be an oriented line segment from A to A' , γ_3 : from B' to B , and γ_2, γ_4 - two arcs of circle γ , while Γ_1, Γ_2 - two parts of Γ .

Then: $\Gamma_1 + (-\gamma_3) + (-\gamma_2) + (-\gamma_1)$ is a closed curve and $\frac{1}{z-a}$ is analytic on it & inside, hence:

$$\int_{\Gamma_1} \frac{dz}{z-a} = \int_{\gamma_1 + \gamma_2 + \gamma_3} \frac{dz}{z-a}$$

Likewise, $\frac{1}{z-a}$ is analytic on & inside of $\Gamma_2 + \gamma_1 + (-\gamma_4) + \gamma_3 \equiv$

$$\Rightarrow \int_{\Gamma_2} \frac{dz}{z-a} = \int_{(-\gamma_3) + \gamma_4 + (-\gamma_1)} \frac{dz}{z-a}$$

$$\text{Thus: } \int_{\Gamma} \frac{dz}{z-a} = \int_{\Gamma_1 + \Gamma_2} \frac{dz}{z-a} = \underbrace{\left(\int_{\gamma_1} \frac{dz}{z-a} + \int_{-\gamma_1} \frac{dz}{z-a} \right)}_{=0} + \underbrace{\left(\int_{\gamma_3} \frac{dz}{z-a} + \int_{-\gamma_3} \frac{dz}{z-a} \right)}_{=0} + \underbrace{\int_{\gamma_2 + \gamma_4 = \gamma} \frac{dz}{z-a}}_{=2\pi i} = 2\pi i$$

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We can summarize the previous exercise as:

$$\int_{\Gamma} \frac{dz}{z-a} = \begin{cases} 0 & \text{if } a \text{ is outside } \Gamma \\ 2\pi i & \text{if } a \text{ is inside } \Gamma \end{cases}$$

$\nwarrow$ positively oriented

This formula is key to many other calculations!

Ex 3: Evaluate $\int_{\Gamma} \frac{z}{(z+2)(z-1)} dz$ where Γ is an ellipse $\frac{x^2}{9} + \frac{y^2}{16} = 1$ oriented counterclockwise.

Step 1: Partial fraction decomposition

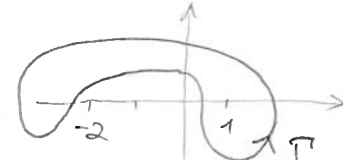
$$\frac{z}{(z+2)(z-1)} = \frac{A}{z+2} + \frac{B}{z-1} \Rightarrow \begin{cases} A = 2/3 \\ B = 1/3 \end{cases}$$

$$\Rightarrow \int_{\Gamma} \frac{z}{(z+2)(z-1)} dz = \frac{2}{3} \int_{\Gamma} \frac{dz}{z+2} + \frac{1}{3} \int_{\Gamma} \frac{dz}{z-1}$$

Step 2: Apply Ex 2 to each term.

Note that both 1, -2 lie inside of $\Gamma \Rightarrow \int_{\Gamma} \frac{dz}{z+2} = 2\pi i = \int_{\Gamma} \frac{dz}{z-1}$

So: We get $\frac{2}{3} \cdot 2\pi i + \frac{1}{3} \cdot 2\pi i = \boxed{2\pi i}$

Ex 4: Evaluate $\int_{\Gamma} \frac{z}{(z+2)(z-1)} dz$ with Γ being 

By previous exercise: $\frac{z}{(z+2)(z-1)} = \frac{2}{3} \cdot \frac{1}{z+2} + \frac{1}{3} \cdot \frac{1}{z-1} \Rightarrow$

However, while 1 is inside Γ , -2 is actually outside!

$$\Rightarrow \int_{\Gamma} \frac{z}{(z+2)(z-1)} dz = \frac{2}{3} \cdot 0 + \frac{1}{3} \cdot 2\pi i = \underline{\underline{\frac{2\pi i}{3}}}$$

This strategy allows to integrate rational functions over any closed Γ . But how about more complicated functions, e.g. $\frac{e^{z^2}}{z-1}$?

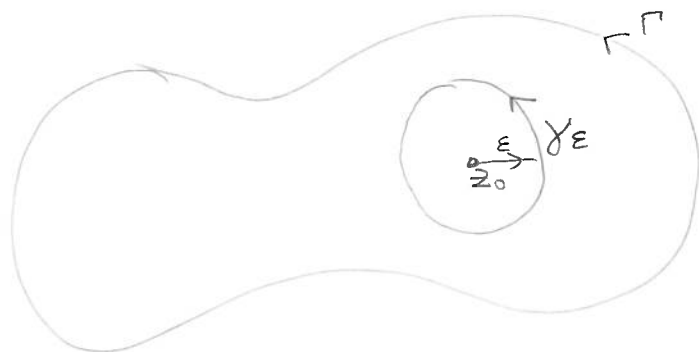
Theorem 2: Let Γ be a simple closed contour positively oriented. If f is analytic on Γ & inside it, and z_0 is any point inside Γ

$$\Rightarrow \boxed{\frac{1}{2\pi i} \int_{\Gamma} \frac{f(z)}{z-z_0} dz = f(z_0)} \leftarrow \text{"Cauchy's Integral Formula"}$$

Proof is quite elementary and is similar to our previous arguments

Step 1: As in Ex 2, we see that $\int_{\Gamma} \frac{f(z)}{z-z_0} dz = \int_{\gamma_{\epsilon}} \frac{f(z)}{z-z_0} dz$ for any $\epsilon > 0$.

Here γ_{ϵ} denotes a radius ϵ circle centered at z_0 , such that it lies inside Γ .



Know: $\int_{\gamma_{\epsilon}} \frac{dz}{z-z_0} = 2\pi i \Rightarrow f(z_0) = \frac{1}{2\pi i} \int_{\gamma_{\epsilon}} \frac{f(z)}{z-z_0} dz$

Step 2: Prove $\int_{\gamma_{\epsilon}} \frac{f(z)}{z-z_0} dz \xrightarrow{\epsilon \rightarrow 0} \int_{\gamma_{\epsilon}} \frac{f(z_0)}{z-z_0} dz$.

Indeed, using estimates from last time, we get:

$$\begin{aligned} \left| \int_{\gamma_{\epsilon}} \frac{f(z) - f(z_0)}{z-z_0} dz \right| &\leq \max_{z \in \gamma_{\epsilon}} \left| \frac{f(z) - f(z_0)}{z-z_0} \right| \cdot \text{length}(\gamma_{\epsilon}) = \\ &= \max_{z \in \gamma_{\epsilon}} |f(z) - f(z_0)| \cdot \frac{1}{\epsilon} \cdot 2\pi\epsilon \end{aligned}$$

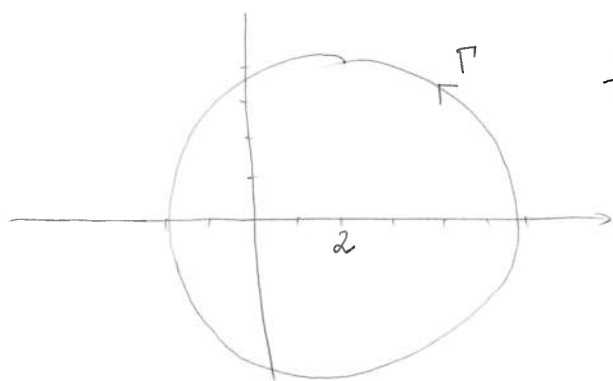
But f - analytic $\Rightarrow f$ - continuous at $z_0 \Rightarrow \max_{z \in \gamma_{\epsilon}} |f(z) - f(z_0)| \xrightarrow{\epsilon \rightarrow 0} 0$

This completes the proof!

Surprising consequence:

Fact: If Γ is a simple closed curve and $f(z)$ is analytic on Γ and inside it, then values of f on Γ determine all values inside Γ !

Ex 5: $\int_{\Gamma} \frac{e^{z^2}}{z-1} dz = ?$, where Γ is a positively oriented circle $|z-2|=4$.



$f(z) = e^{z^2}$ is an entire function?
 1 is inside Γ

$$\int_{\Gamma} \frac{e^{z^2}}{z-1} dz = 2\pi i \cdot f(1) = \underline{\underline{2\pi i \cdot e}}$$