Lecture #16

Ex 1: Evaluate $\int_\Gamma \frac{e^{z^2}}{z^2-4}dz$ when:

a) Γ is the unit circle $|z|=1$ oriented counterclockwise

b) Γ is the circle $|z|=3$ positively oriented

c) Γ is the circle $|z-1|=2$ positively oriented.

a) $\frac{e^{z^2}}{z^2-4}$ is analytic on unit disk $|z|\le 1 \Rightarrow \int_\Gamma \frac{e^{z^2}}{z^2-4}dz = 0$

b) As $z^2-4=(z-2)(z+2)$ and both ± 2 are inside Γ we need to be more careful now and shall rather use Cauchy's Integral Formula

Partial fraction decomposition: $\frac{1}{z^2-4} = \frac{1}{4}\cdot\frac{1}{z-2} - \frac{1}{4}\cdot\frac{1}{z+2}$

$$\Rightarrow \frac{1}{4}\underbrace{\int_\Gamma \frac{e^{z^2}}{z-2}dz}_{=2\pi i\cdot e^{z^2}|_{z=2}} - \frac{1}{4}\underbrace{\int_\Gamma \frac{e^{z^2}}{z+2}dz}_{=2\pi i\cdot e^{z^2}|_{z=-2}} = \frac{2\pi i}{4}(e^4-e^4)=0.$$

c) Only 2 is contained inside Γ (while -2 is outside!)

$$\Rightarrow \frac{1}{4}\int \frac{e^{z^2}}{z-2}dz - \frac{1}{4}\int_\Gamma \frac{e^{z^2}}{z+2}dz = \frac{2\pi i e^4}{4} = \frac{\pi i e^4}{2}.$$

Note: In part c) one could also argue by rather viewing $\frac{e^{z^2}}{z^2-4}$ as $\frac{e^{z^2}/(z+2)}{z-2}$ and as $\frac{e^{z^2}}{z+2}$ is analytic inside $\Gamma \Rightarrow$ get $2\pi i\cdot\left(\frac{e^{z^2}}{z+2}\right)\Big|_{z=2} = \frac{\pi i e^4}{2}$.

Def: A domain D is called simply-connected (a.k.a. "without holes") if any simple closed contour lying in D has interior wholly in D

Ex 2: Which of the following are simply-connected:

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Main result last time was the Cauchy's Integral formula:

$$\boxed{f(z) = \frac{1}{2\pi i}\int_\Gamma \frac{f(w)}{w-z}\,dw} \leftarrow$$

- Γ is any positively oriented simple closed curve
- f is analytic inside & on Γ
- z is inside Γ.

As the right-hand side only involves values of f on Γ, we get a quite surprising result (with assumptions as above):

Corollary: Values of f on Γ determine values of f inside Γ!

Looking at the top boxed formula, and assuming derivative w.r.t. z and integration w.r.t. dw can be swapped, we arrive at guess:

$$f'(z) = \frac{1}{2\pi i}\int_\Gamma \left[\frac{d}{dz}\left(\frac{f(w)}{w-z}\right)\right]dw = \frac{1}{2\pi i}\int_\Gamma \frac{f(w)}{(w-z)^2}\,dw.$$

We shall now prove this rigorously in a bigger generality:

Theorem 1: Let g be a continuous function on a contour Γ. For each $z \notin \Gamma$, consider $G(z) := \int_\Gamma \frac{g(w)}{w-z}\,dw$ (well-defined as $\frac{g(w)}{w-z}$ is continuous!)

Then G is analytic at each $z \notin \Gamma$ and

$$G'(z) = \int_\Gamma \frac{g(w)}{(w-z)^2}\,dw$$

When Γ is a positively oriented simple closed curve we get above

$$\boxed{f'(z) = \frac{1}{2\pi i}\int_\Gamma \frac{f(w)}{(w-z)^2}\,dw \quad \text{for any } z \text{ inside } \Gamma}$$

► By definition, we have

$$G'(z) = \lim_{\Delta z \to 0} \frac{G(z+\Delta z) - G(z)}{\Delta z} = \lim_{\Delta z \to 0}\left\{\frac{1}{\Delta z}\int_\Gamma \left(\frac{g(w)}{w-z-\Delta z} - \frac{g(w)}{w-z}\right)dw\right\}$$

$$= \lim_{\Delta z \to 0}\int_\Gamma \frac{g(w)}{(w-z)(w-z-\Delta z)}\,dw$$

Want: $\displaystyle\int_\Gamma \frac{g(w)}{(w-z)(w-z-\Delta z)}\,dw \xrightarrow[\Delta z \to 0]{} \int_\Gamma \frac{g(w)}{(w-z)^2}\,dw$

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(continuation of proof)

$$\int_\Gamma \frac{g(w)}{(w-z)(w-z-\Delta z)}dw - \int_\Gamma \frac{g(w)}{(w-z)^2}dw = \int_\Gamma \frac{g(w)\cdot \Delta z}{(w-z)^2(w-z-\Delta z)}dw$$

Remarks: $\displaystyle\int_\Gamma \frac{g(w)\cdot\Delta z}{(w-z)^2(w-z-\Delta z)}dw \xrightarrow[\Delta z\to 0]{} 0$

But as $z\notin\Gamma$, there is some $\rho\in\mathbb{R}_{>0}$ such that $|w-z|\geq\rho$ for any $w\in\Gamma$.
Hence, if $|\Delta z|<\rho/2$, we get $\left|\frac{1}{(w-z)^2(w-z-\Delta z)}\right| \leq \frac{1}{\rho^2\cdot\rho/2} = \frac{2}{\rho^3}$.

Also, as g is continuous on Γ, it's bounded: $|g(w)|\leq M$ for all $w\in\Gamma$
some real

So: $\displaystyle\left|\int_\Gamma \frac{g(w)\,\Delta z}{(w-z)^2(w-z-\Delta z)}dw\right| \leq \underbrace{|\Delta z|\cdot \underbrace{M\cdot\frac{2}{\rho^3}\cdot \text{length}(\Gamma)}_{\text{constant}}}_{\to 0 \text{ as } |\Delta z|\to 0}$

∎

Same argument can be also applied to prove:

> **Theorem 1'**: Let g be a continuous function on a contour Γ. For each $z\notin\Gamma$ consider $G(z):=\int_\Gamma \frac{g(w)}{(w-z)^n}dw$. Then G is analytic at each $z\notin\Gamma$ and
> $$G'(z)=n\cdot\int_\Gamma \frac{g(w)}{(w-z)^{n+1}}dw$$

In the particular case when Γ is a positively oriented simple closed contour and z is inside Γ, we get:

> **Theorem 2** (higher order Cauchy theorem): If f is analytic on & inside Γ, then
> $$\boxed{f^{(n)}(z) = \frac{1}{2\pi i}\cdot n!\cdot\int_\Gamma \frac{f(w)}{(w-z)^{n+1}}dw}$$
> for any z inside Γ, where $n! = 1\cdot 2\cdot 3\cdot\ldots\cdot n$

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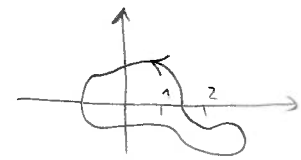
We shall often apply this result written in the form:

$$\boxed{\int_\Gamma \frac{f(w)}{(w-z)^{n+1}}\,dw = \frac{2\pi i}{n!} f^{(n)}(z)}$$

Ex 3: Evaluate $\int_\Gamma \frac{e^{z^2}}{z^2(z-2)}dz$ with contour Γ on



Note that while 0 is inside Γ, 2 is outside!

Write $\frac{e^{z^2}}{z^2(z-2)} = \frac{e^{z^2}/(z-2)}{z^2}$, and apply above formula with $n=1$

and $f(z) = \frac{e^{z^2}}{z-2}$ which is analytic on & inside Γ to get:

$$\int_\Gamma \frac{e^{z^2}}{z^2(z-2)}dz = 2\pi i \cdot f'(0) = 2\pi i \cdot \left.\frac{e^{z^2}\cdot 2z(z-2) - e^{z^2}\cdot 1}{(z-2)^2}\right|_{z=0} = -\frac{\pi i}{2}$$

Let's summarize several corollaries of previous theorems.

Theorem 3: If f is analytic in a domain D, then all its derivatives $f^{(n)}$ exist and are analytic in D!

Note: this is very different from usual calculus case!

Theorem 4: If $f = u + iv$ is analytic in a domain D, then all partials of u, v exist & continuous in D (i.e. u, v – C^∞-smooth)

Finally, in combination with Lecture 14, we get:

Theorem 5 ("Morera's theorem"): If f is continuous in a domain D and $\int_\Gamma f(z)dz = 0$ for every closed contour in D, then f is analytic!

By Lecture 14, above implies that f has antiderivative F, i.e. $F' = f$. Thus F is analytic, hence so is $f = F'$ by Theorem 3!

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Let's end with the following important estimate (to be used next time).

Lemma ("Cauchy Estimates"): Let f be analytic on & inside a circle $C_R(z_0)$ (centered at $z_0 \in \mathbb{C}$ of radius R). If $|f(z)| \leq M$ for all $z \in C_R(z_0)$, then
$$|f^{(n)}(z_0)| \leq \frac{n! \cdot M}{R^n} \quad \text{for } n = 1, 2, 3, \ldots$$

► By Theorem 2:
$$|f^{(n)}(z_0)| = \left| \int_{C_R(z_0)} \frac{f(w)}{(w-z_0)^{n+1}} \, dw \right| \cdot \frac{n!}{2\pi} \leq \frac{M}{R^{n+1}} \cdot \underbrace{\text{length}(C_R(z_0))}_{=2\pi R} \cdot \frac{n!}{2\pi} = \frac{n! \cdot M}{R^n}$$
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