Lecture #17

- Last time: many foundational results such as:
  - Morera's theorem
  - Higher Order Cauchy Integral Formula

$$\boxed{\int_\Gamma \frac{f(z)}{(z-z_0)^{n+1}}dz = \frac{2\pi i}{n!} f^{(n)}(z_0)}$$

Γ - positively oriented simple closed curve
z_0 - point inside Γ
f - analytic inside Γ

Ex1: Evaluate $I = \int_\Gamma \frac{e^{z^2}}{(z-a)^2(z-b)^2}dz$ where Γ as above. (← Could also do via partial fraction decomposition)

Case 1: a, b - outside Γ $\Rightarrow I = 0$ by Cauchy's theorem.

Case 2: a - inside Γ, b - outside $\Rightarrow$ by above formula (with $n=1$):

$$I = \int_\Gamma \frac{e^{z^2}/(z-b)^2}{(z-a)^2}dz = \frac{2\pi i}{1!}\left(\frac{e^{z^2}}{(z-b)^2}\right)'\Big|_{z=a} = 2\pi i \cdot \frac{e^{z^2}\cdot 2z(z-b)^2 - e^{z^2}\cdot 2(z-b)}{(z-b)^4}\Big|_{z=a}$$

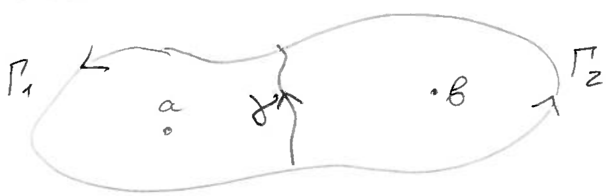
$$= \frac{2\pi i \cdot e^{a^2}\cdot 2\,(a(a-b)-1)}{(a-b)^3}$$

Case 3: a - outside, b - inside Γ $\Rightarrow$ same formula with $a \leftrightarrow b$, i.e.

$$I = \frac{4\pi i\, e^{b^2}(b(b-a)-1)}{(b-a)^3}$$

Case 4: $a = b$ - inside $\Rightarrow I = \int_\Gamma \frac{e^{z^2}}{(z-a)^4}dz = \frac{2\pi i}{6}\left(e^{z^2}\right)'''\Big|_{z=a}$ ← finish at home

Case 5: $a \neq b$ - inside



Writing $\int_\Gamma = \int_{\Gamma_1+\gamma} + \int_{(-\gamma)+\Gamma_2}$, we see that the answer is actually the sum of above answers in Case 2 & 3

$$I = \frac{4\pi i}{(a-b)^3}\left(e^{a^2}(a^2-ab-1) - e^{b^2}(b^2-ab-1)\right)$$

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Recall Cauchy's Estimates from last time:

$$\boxed{|f^{(n)}(z_0)| \leq \frac{n! \cdot M}{R^n} \quad \text{for any } n=1,2,3,\ldots}$$

where f - analytic inside and on circle $C_R(z_0)$ and $|f(z)| \leq M$ on $C_R(z_0)$

Theorem 1 (Liouville Theorem): The only bounded (i.e. $|f(z)| \leq M$ for all z) entire functions are constants

▸ Fix any $z_0 \in \mathbb{C}$, then for any $R \in \mathbb{R}_{\geq 0}$: $|f'(z_0)| \leq \frac{M}{R} \xrightarrow[R \to +\infty]{} 0$

Thus $f'(z) \equiv 0 \Rightarrow f(z) \equiv \text{constant}$ ∎

[Note: There is no such statement in usual calculus over $\mathbb{R}$!

Theorem 2 (Fundamental Theorem of Algebra): Every nonconstant polynomial with $\mathbb{C}$ coefficients has at least one zero

[Note: As discussed earlier this implies deg n pol-s have n complex roots!

▸ Assume contrary: let $p(z) = a_0 + a_1 z + \ldots + a_n z^n$ with $n \neq 0$ have no roots.

Consider $f(z) = 1/p(z)$ which is then entire! Due to Liouville theorem it remains to show $f(z)$ is bounded on whole $\mathbb{C}$.

As $\frac{p(z)}{z^n} \xrightarrow[z \to \infty]{} a_n$, it means there is some $R > 0$ such that

$\left|\frac{p(z)}{z^n}\right| > \frac{|a_n|}{2}$ whenever $|z| > R \Rightarrow |f(z)| \leq \frac{2}{|a_n| \cdot R^n}$ for $|z| > R$.

On the other hand, $f(z)$ is continuous on closed bounded disk $D_R(0) \Rightarrow |f(z)|$ is bounded by some constant M on $D_R(0)$.

So: $|f(z)| \leq \max\{M, \frac{2}{|a_n| R^n}\}$ for all $z \in \mathbb{C}$ ∎

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Let's rewrite Cauchy integral formula for $\Gamma = C_R(z_0)$ – circle of radius R centered at z_0

$$f(z_0) = \frac{1}{2\pi i} \int_{C_R(z_0)} \frac{f(z)}{z - z_0}\, dz \underset{\substack{z(t) = z_0 + Re^{it} \\ 0 \le t \le 2\pi}}{\overset{\text{parametrize}}{=\!=\!=}} \frac{1}{2\pi i} \int_0^{2\pi} \frac{f(z_0 + Re^{it})}{Re^{it}} \cdot iRe^{it}\, dt$$

$$\Downarrow$$

$$\boxed{f(z_0) = \frac{1}{2\pi} \int_0^{2\pi} f(z_0 + Re^{it})\, dt} \quad \leftarrow \text{"mean-value property"}$$

Lemma 1: Let f be an analytic function on the disk $D_r(z_0)$.
If $\max_{z \in D_r(z_0)} |f(z)| = |f(z_0)|$, then $|f(z)| \equiv$ constant on $D_r(z_0)$

The proof is based on the following simple fact from calculus:

Fact: If $h: [a,b] \to \mathbb{R}$ is continuous, $h(t) \le M$, $\int_a^b h(t)\,dt = M(b-a)$
then $h(t) = M$ for all $t \in [a,b)$

▶ If Lemma 1 fails then there is $z_1 \in D_r(z_0)$ with $|f(z_1)| < |f(z_0)|$
It remains to apply above Fact and mean value property for $R = |z_1 - z_0|$ ∎

This can be easily generalized to

Theorem 3: If f is analytic in domain Ω and $|f(z)|$ achieves maximum in an interior point z_0 in Ω, then $f \equiv$ const in Ω

▶ First, we note that it suffices to show $|f| \equiv$ const by earlier discussion
Assume there is $z_1 \in \Omega$ with $|f(z_0)| > |f(z_1)|$. Connect z_0 & z_1 by a path γ, parametrized via $z: [0,1] \to \gamma$ with $\begin{cases} z(0) = z_0 \\ z(1) = z_1 \end{cases}$

By Lemma 1 above, $|f(z(t))| = |f(z_0)|$ for all $0 \le t < \varepsilon$ with some $\varepsilon > 0$

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(Continuation of proof)

Let $\tau := \sup\{ t \mid |f(z(s))| = |f(z_0)| \text{ for all } 0 \le s \le t\}$.

As $|f(z(t))|$ is continuous, we actually see $\sup\{\dots\} = \max\{\dots\}$

We claim $\tau = 1$, which would then imply $|f(z_1)| = |f(z_0)| \Rightarrow$ ↯.

If $\tau < 1$, then by Lemma 1 applied to $z(\tau)$ instead of z_0, we see that there is some $\varepsilon > 0$ s.t. $|f(z(t))| = |f(z_0)|$ for $t \le \tau + \varepsilon$, which contradicts definition of τ! ◻

Corollary 1 ("Maximum modulus principle"): A function analytic in a bounded domain Ω and continuous including its boundary attains its maximum modulus on the boundary of Ω

- Ex 2: Is there a minimum modulus principle?

  ▸ Clearly not, e.g. $f(z) = z$ and $\Omega = D_1(0)$ ◻

- Ex 2': Is there a minimum modulus principle for nowhere vanishing analytic $f(z)$?

  ▸ Yes: follows from max modulus principle for $\frac{1}{f(z)}$ ◻

Finally, note that if $\underset{x+iy}{f(z)} = u(x,y) + i \cdot v(x,y)$, then taking real parts of both sides in the "mean-value property" we get:

$$\boxed{u(z_0) = \frac{1}{2\pi}\int_0^{2\pi} u(z_0 + Re^{it})\,dt} \leftarrow \text{"mean-value property"}$$

Corollary 2: Harmonic functions $u(x,y)$ on $D_R(z_0)$ satisfy above!

▸ Follows as $u(x,y)$ is a real part of some analytic $f(z)$ ◻