Lecture #18

• Last time:

1) Liouville's theorem: any bounded entire function is constant
$\Downarrow$
2) Fundamental theorem of algebra: any non-constant polynomial has roots/$\mathbb{C}$
(key idea: given a nowhere vanishing polynomial $p(z)$, apply Liouville's thm to $1/p(z)$)

3) Mean-value property (assuming f is analytic on $D_R(z_0)$)

$$\boxed{f(z_0) = \frac{1}{2\pi}\int_0^{2\pi} f(z_0 + Re^{it})\,dt}$$

$\Downarrow$

4) An analytic function f on any domain D and its boundary, s.t. $|f(z)|$ achieves max in a (interior) point of D is constant
$\Downarrow$
5) Maximum modulus principle

Ex 1: Prove that if $f(z)$ - entire function with $\operatorname{Re} f(z) \leq M$ for all z, then $f \equiv$ constant

► Consider $g(z) = e^{f(z)}$. Then $g(z)$ - entire, and bounded $|g(z)| = e^{f(z)} \leq e^M$.
Hence by Liouville's theorem, $e^{f(z)} \equiv$ constant $\underset{f\text{-continuous}}{\Longrightarrow}$ $f(z)$ - constant ∎

Ex 2: Find $\max\limits_{|z|\leq 1} |az^n + b|$.

► By triangular inequality $|az^n + b| \leq |a|\cdot|z|^n + |b| \leq |a| + |b|$.
The above bound $|a|+|b|$ is indeed max: take z on the unit circle so that $az^n \in \mathbb{R}_{>0}\cdot b$, e.g. take $z = e^{i\theta}$ with $n\theta = \operatorname{Arg} b - \operatorname{Arg} a$
(if a or b are zero, the answer is obvious) ∎

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Ex3: Find $\max_{|z|\le 1} |(z-1)(z+\frac{1}{2})|$

As $(z-1)(z+\frac{1}{2}) = z^2 - \frac{1}{2}z - \frac{1}{2}$ is analytic on $D_1(0)$, by max modulus principle max is achieved on the boundary, i.e. on $C_1(0)$. We shall now parametrize $C_1(0)$ and reduce the question to calculus problem.

$z : [0, 2\pi] \longrightarrow C_1(0)$, $t \mapsto e^{it}$ with $0 \le t \le 2\pi$.

Note that then $|(z-1)(z+\frac{1}{2})|^2 = (e^{2it} - \frac{1}{2}e^{it} - \frac{1}{2})\overline{(e^{2it} - \frac{1}{2}e^{it} - \frac{1}{2})} =$

$= (e^{2it} - \frac{1}{2}e^{it} - 1/2)(e^{-2it} - \frac{1}{2}e^{-it} - \frac{1}{2}) = 1 - \frac{1}{2}e^{it} - \frac{1}{2}e^{2it} - \frac{1}{2}e^{-it} + \frac{1}{4} + \frac{1}{4}e^{it} - \frac{1}{2}e^{-2it} + \frac{1}{4}e^{-it} + \frac{1}{4}$

$= \frac{3}{2} - \frac{1}{2}(e^{2it} + e^{-2it}) - \frac{1}{4}(e^{it} + e^{-it}) = \boxed{\frac{3}{2} - \cos(2t) - \frac{1}{2}\cos t}$

So now we need to find $\max_{0\le t\le 2\pi} g(t)$ with $g(t) = \frac{3}{2} - \cos(2t) - \frac{1}{2}\cos t$

Note: $g(0) = g(2\pi) = \frac{3}{2} - 1 - \frac{1}{2} = 0$

Also look at $g'(t) = 0$, i.e. $0 = \sin(2t)\cdot 2 + \frac{1}{2}\sin t = 4\sin t\cos t + \frac{1}{2}\sin t$

$\Rightarrow \sin t = 0$ or $\cos t = -1/8$.

- $\sin t = 0 \Rightarrow t = 0, \pi, 2\pi$. (already treated $0, 2\pi$)

  $g(\pi) = \frac{3}{2} - 1 + \frac{1}{2} = 1$

- $\cos t = -\frac{1}{8} \Rightarrow \cos(2t) = 2\cos^2 t - 1 = \frac{2}{64} - 1 = -\frac{31}{32}$

  $\Rightarrow g(t) = \frac{3}{2} + \frac{31}{32} + \frac{1}{16} = \frac{81}{32} > 1$

So: $\max_{0\le t\le 2\pi} g(t) = \frac{81}{32} \Rightarrow \max_{|z|\le 1} |(z-1)(z+\frac{1}{2})| = \sqrt{\frac{81}{32}} = \frac{9\sqrt{2}}{8}$ ∎

Lecture #18 $\boxed{\S 4.7}$

We shall now generalize all results from Lecture 17 to harmonic f-s.
First, let's give now a short self-contained proof of [Lecture 8, Thm 2]

Theorem 1: Let $u = u(x,y)$ be a harmonic function on a simply connected domain D. Then there is an analytic $f(z)$ on D with $\mathrm{Re}\, f = u$

in other words, u has a harmonic conjugate.

▸ If $f = u + iv$ is such an analytic f-n, then $f' = \begin{cases} u_x + i\cdot v_x \\ v_y - i\cdot u_y \end{cases}$ and evoking CR equations we can write $f' = u_x - iu_y$. This gives us the key idea:

Step 1: Consider $g(z = x+iy) = u_x - iu_y$. Then g is analytic

Indeed, we just need to check CR:

- $u_{xx} = (-u_y)_y$ follows from u being harmonic
- $(u_x)_y = -(-u_y)_x$ follows from equality of mixed partials.

Step 2: $g(z)$ has an antiderivative $F(z)$ on a simply connected D.
Write $F(z) = U + iV$. Comparing $F'(z) = U_x - iU_y$ to $g(z) = u_x - iu_y$,
we get $U_x = u_x$, $U_y = u_y$ $\Rightarrow$ $U - u = c$ - constant.

Step 3: Function $f(z) := F(z) - c$ is the desired one. ∎

Theorem 2 (Liouville's theorem for harmonic functions): If $u(x,y)$ is harmonic on $\mathbb{R}^2$ and bounded $u(x,y) \le M$ for all x,y, then $u \equiv \mathrm{const}$

▸ As $\mathbb{C} = \mathbb{R}^2$ is simply connected, by Thm 1 we have an entire f-n $f = u + iv$. Then $u = \mathrm{Re}\, f$ and we can apply Ex 1 to conclude that $f \equiv \mathrm{const}$ $\Rightarrow$ $u \equiv$ constant ∎

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Lemma (Mean-value property for harmonic): If u is harmonic on $D_R(z_0)$ then $\boxed{u(z_0) = \frac{1}{2\pi}\int_0^{2\pi} u(z_0 + Re^{it})\,dt}$

► Applying Thm 1, pick an analytic $f(z) = u(z) + iv(z)$ on $D_R(z_0)$. By mean-value property for analytic functions, we have $f(z_0) = \frac{1}{2\pi}\int_0^{2\pi} f(z_0 + Re^{it})\,dt$. Taking real parts of both sides implies the claim ∎

This result has a number of consequences (as last time):

Corollary 1 (Max principle): A harmonic function on domain D and its boundary attains its maximum on the boundary

Note that if u - harmonic $\Rightarrow$ $-u$ - harmonic, and $\max(-u) = -\min u$. Hence, in contrast to analytic setup, we have min principle:

Corollary 2 (Min principle): A harmonic function on domain D and its boundary attains its minimum on the boundary

Likewise, we have:

Corollary 3: If u is harmonic in a domain D and its boundary and achieves its max or min in a (interior) point of D, then $u \equiv \text{const}$

Combining these results, we obtain:

Theorem 3: Let $u_1(x,y), u_2(x,y)$ be harmonic functions in a bounded domain D and its boundary, such that $u_1 = u_2$ on the boundary of D. Then $u_1 = u_2$ on all D!

► Apply Cor 1 & 2 to harmonic $u_1 - u_2$ which is 0 on boundary of D ∎

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Ex 4: Find all solutions $\phi(x,y)$ of Laplace equation in the washer $1 \le |z| \le 3$ with initial conditions $\phi|_{|z|=1} = 10$, $\phi|_{|z|=3} = 40$.

▶ In [Lecture 11, Ex 4] we found one such solution:

$$\phi(x,y) = \frac{30}{\ln 3} \log\sqrt{x^2+y^2} + 10.$$

As washer is bounded, above thm implies this solution is unique! ◼

Ex 5: Is theorem 3 true for unbounded domains?

▶ No. Let $D = \{(x,y) \mid y \ge 0\}$ - upper half plane. Then y, $2y$ are both harmonic and both vanish on the boundary of D, i.e. $\{(x,0) \mid x \in \mathbb{R}\}$ ◼

In general, solving a Laplace equation $\nabla^2\phi = 0$ on D with initial condition $\phi|_{\partial D = \text{boundary of } D}$ given is called Dirichlet Problem

$\boxed{\text{Q: Can one write explicit solution of Dirichlet Problem analogously to Cauchy integral formula?}}$

Recall: If f is analytic in $D_R(0)$, then for any $|z| < R$:

$$\boxed{2\pi i \cdot f(z) = \int_{C_R(0)} \frac{f(w)}{w-z}\, dw}$$

However, in contrast to the mean-value property taking real parts is not immediate.

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Trick: Use $\displaystyle\int_{C_R(0)} \frac{f(w)\bar{z}}{R^2 - w\bar{z}}\,dw = 0$ as $\dfrac{f(w)\cdot\bar{z}}{R^2 - w\bar{z}}$ is an analytic function of w on & inside $C_R(0)$.

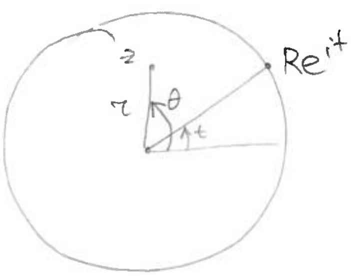
$$\Downarrow$$

$$f(z) = \frac{1}{2\pi i}\int_{C_R(0)} \left(\frac{f(w)}{w-z} + \frac{f(w)\bar{z}}{R^2 - w\bar{z}}\right)dw = \frac{1}{2\pi i}\int_{C_R(0)} \frac{R^2 - z\cdot\bar{z}}{(w-z)(R^2 - w\bar{z})} f(w)\,dw$$

Let's compute the latter integral directly by parametrizing $C_R(0)$ via $w = Re^{it}$, $0 \le t \le 2\pi$:

$$f(z) = \frac{1}{2\pi i}\int_0^{2\pi} \frac{R^2 - |z|^2}{(Re^{it} - z)(R^2 - Re^{it}\bar{z})} f(Re^{it})\cdot iRe^{it}\,dt$$

$$= \frac{R^2 - |z|^2}{2\pi}\int_0^{2\pi} \frac{f(Re^{it})}{(Re^{it} - z)(Re^{-it} - \bar{z})}\,dt = \boxed{\frac{R^2 - |z|^2}{2\pi}\int_0^{2\pi} \frac{f(Re^{it})}{|Re^{it} - z|^2}\,dt}$$



Finally if $z = re^{i\theta}$, then by the law of cosines:

$$\boxed{|Re^{it} - z|^2 = R^2 + r^2 - 2rR\cos(\theta - t)}$$

This implies the following fundamental property (by taking Re)

Theorem 4 (Poisson integral formula): If ϕ is harmonic in $D_R(0)$, then

$$\phi(re^{i\theta}) = \frac{R^2 - r^2}{2\pi}\int_0^{2\pi} \frac{\phi(Re^{it})}{R^2 + r^2 - 2rR\cos(\theta - t)}\,dt$$

$\uparrow$ this is an explicit solution of the Dirichlet Problem on a disk

(the function $\dfrac{R^2 - r^2}{2\pi(R^2 + r^2 - 2rR\cos(\theta - t))}$ is called Poisson kernel)