Lecture #21

Def: A series $\sum_{j=0}^{\infty} c_j$ converges to L if partial sums $S_n \xrightarrow[n\to\infty]{} L$, where

$$S_n = \sum_{j=0}^{n} c_j$$

[Note: If $\sum_{j=0}^{\infty} c_j$ converges, then $c_j \to 0$ as $j \to \infty$

(as $c_j = S_j - S_{j-1} \xrightarrow[j\to\infty]{} L - L = 0$)

Ex 1: For which $z \in \mathbb{C}$, is $\sum_{j=0}^{\infty} z^j$ converging?

$$S_n = 1 + z + z^2 + \_\_ + z^n = \frac{1-z^{n+1}}{1-z} = \frac{1}{1-z} - \frac{z^{n+1}}{1-z}$$

Case 1: If $|z|<1 \Rightarrow z^{n+1} \xrightarrow[n\to\infty]{} 0 \Rightarrow \frac{z^{n+1}}{1-z} \xrightarrow[n\to\infty]{} 0 \Rightarrow S_n \longrightarrow \frac{1}{1-z}$

Case 2: If $|z| \geq 1 \Rightarrow |z^n| \geq 1 \Rightarrow$ diverges!

So: Converges $\Leftrightarrow |z|<1$

Couple of familiar Results from usual calculus:

"Comparison Test": If $|c_j| < M_j$ for all $j \geq N$ and $\sum_{j=0}^{\infty} M_j$ converges, then $\sum_{j=0}^{\infty} c_j$ converges.

"Ratio Test": Assume $\left|\frac{c_{j+1}}{c_j}\right| \xrightarrow[j\to\infty]{} L \in \mathbb{R}$.

a) If $L<1 \Rightarrow \sum_{j=0}^{\infty} c_j$ converges

b) If $L>1 \Rightarrow \sum_{j=0}^{\infty} c_j$ diverges

[Note: If $L=1$, we cannot conclude anything

Ex 2: Prove that $\sum_{j=1}^{\infty} \frac{1000}{j^j}$ converges

As $\frac{1}{j^j} \leq \frac{1}{2^j}$ for $j \geq 1$ and $\sum_{j=1}^{\infty} \frac{1}{2^j}$ converges by Ex 1, the result follows due to "comparison test"

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Ex 3: Prove that $\sum_{j=1}^{\infty} \frac{1}{j^p}$ converges for $p \in \mathbb{R}_{>1}$.

▸ As $\frac{1}{j^p} \le \int_{j-1}^{j} \frac{dx}{x^p}$ and $\int_{1}^{\infty} \frac{dx}{x^p}$ is finite for $p>1$, we get the result by comparison test ∎

Ex 4: Show that $\sum_{j=1}^{\infty} \frac{j^2}{4^j}$ converges.

▸ If $c_j = \frac{j^2}{4^j} \Rightarrow \frac{c_{j+1}}{c_j} = \frac{(j+1)^2}{j^2 \cdot 4} = \frac{1}{4}\left(1+\frac{1}{j}\right)^2 \xrightarrow[j\to\infty]{} \frac{1}{4} < 1 \Rightarrow$ converges by ratio test ∎

Def: $\sum_{j=0}^{\infty} c_j$ is *absolutely convergent* if $\sum_{j=0}^{\infty} |c_j|$ converges

[Note: Absolute convergence $\overset{\Rightarrow}{\not\Leftarrow}$ convergence

Def: a) The sequence $\{f_n(z)\}$ *converges uniformly* to $f(z)$ on D if $\forall \varepsilon > 0\ \exists N$ such that $|f_n(z) - f(z)| < \varepsilon \quad \forall z \in D,\ \forall n \ge N$

b) The series $\sum_{j=0}^{\infty} f_j(z)$ *converges uniformly* to $f(z)$ on D if $\sum_{j=0}^{n} f_j(z) \longrightarrow f(z)$ uniformly on D.

Examples: a) $z^n \longrightarrow 0$ uniformly on $D = D_r(0)$ for any $0 < r < 1$
(as $|z^n| \le r^n \xrightarrow[n\to\infty]{} 0$)

b) z^n does not converge uniformly to 0 on $D_1(0) = \{z : |z| < 1\}$
(as $\lim_{z \to 1-} |z^n| = 1$)

c) $\sum_{j=0}^{\infty} z^j \longrightarrow \frac{1}{1-z}$ uniformly on $D = D_r(0)$ for any $r<1$
(as $\left|S_n(z) - \frac{1}{1-z}\right| = \left|\frac{z^{n+1}}{1-z}\right| \le \frac{r^{n+1}}{1-r} \xrightarrow[n\to\infty]{} 0$)

d) $\sum_{j=0}^{\infty} z^j$ does not converge uniformly to $\frac{1}{1-z}$ on $D_1(0)$.

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The following result will be proved later:

Fact: For any power series $\sum_{j=0}^{\infty} a_j (z-z_0)^j$ there is $0 \le R \le +\infty$ s.t.

a) above series converges for $|z-z_0|<R$

b) above series diverges for $|z-z_0|>R$

c) above series converges uniformly on $D_r(z_0)$ for any $r<R$

Such R is called radius of convergence (R.O.C)

Example: As follows from Ex 1, R.O.C of $\sum_{j=0}^{\infty} z^j$ is $R=1$.

Ex 5: Find radius of convergence of $\sum_{j=1}^{\infty} j\, 5^j z^j$.

▶ By ratio test, above series converges is $|5z|<1$ and diverges if $|5z|>1$. Hence, R.O.C. is $R=1/5$ ◼

Ex 6: Find radius of convergence of $\sum_{j=0}^{\infty} \frac{z^j}{j!}$

▶ By ratio test, see $R=+\infty$ ◼

Def: If f is analytic at z_0, then the Taylor series of f around z_0 is:

$$f(z_0)+\frac{f'(z_0)}{1!}(z-z_0)+\frac{f''(z_0)}{2!}(z-z_0)^2+\ldots=\sum_{n=0}^{\infty}\frac{f^{(n)}(z_0)}{n!}(z-z_0)^n$$

For $z_0=0$, it's called Maclaurin series.

Theorem 1: a) If f is analytic in $D_R(z_0)$ then the Taylor series of f around z_0 converges to f for all $z\in D_R(z_0)$.

b) If f is analytic in $D_R(z_0)$, then the Taylor series of f around z_0 converges uniformly to $f(z)$ on $D_r(z_0)$ for any $r<R$.

We shall give the proof next time.

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Examples:

a) For $f(z)=e^z$, entire function, we get

$$\boxed{\sum_{j=0}^{\infty}\frac{z^j}{j!}=e^z \quad \text{for all } z\in\mathbb{C}}$$

b) For $f(z)=\frac{1}{1-z}=(1-z)^{-1}$, analytic in $D_1(0)$, we get

$f^{(n)}(z)=n!\,(1-z)^{-n-1} \Rightarrow \boxed{\sum_{j=0}^{\infty}z^j=\frac{1}{1-z} \text{ for } |z|<1}$ (see Ex 1)

c) For $f(z)=\operatorname{Log} z$ and $z_0=1$, get $f(z)$ is analytic in $D_1(1)$, and

$f^{(n)}(z)=(-1)^{n-1}(n-1)!\,z^{-n} \Rightarrow \boxed{\sum_{j=1}^{\infty}\frac{(-1)^{j-1}}{j}(z-1)^j=\operatorname{Log} z \quad \text{for } |z-1|<1}$

d) For $f(z)=\sin z$, its Maclaurin series gives $\boxed{z-\frac{z^3}{3!}+\frac{z^5}{5!}-\frac{z^7}{7!}+\ldots=\sin z \text{ for all } z\in\mathbb{C}}$

e) For $f(z)=\cos z$, we likewise get $\boxed{1-\frac{z^2}{2!}+\frac{z^4}{4!}-\frac{z^6}{6!}+\ldots=\cos z \quad \forall z\in\mathbb{C}}$

Note: both sides of e) are obtained by differentiation of d)!

Theorem 2: If f is analytic at z_0, then the Taylor series for f' around z_0 is obtained by termwise differentiation of the Taylor series for f around z_0 and converges in same disk

▶ Follows immediately from $(f')^{(n)}=f^{(n+1)}$ and the fact that if f - analytic in $D_R(z_0)$, then so is f'. ∎

Claim: If $f(z)=\sum_{j=0}^{\infty}a_j(z-z_0)^j$, $g(z)=\sum_{j=0}^{\infty}b_j(z-z_0)^j$ are Taylor series then Taylor series of $f+g$ and $c\cdot f$ around z_0 are:

$$(f+g)(z)=\sum_{j=0}^{\infty}(a_j+b_j)(z-z_0)^j, \quad c\cdot f(z)=\sum_{j=0}^{\infty}c\,a_j(z-z_0)^j$$