Lecture #22

Today we shall start by proving two basic results about Taylor series and power series stated last time.

- Recall that if $f(z)$ is analytic near z_0, then the Taylor series of f around z_0:

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)}{n!}(z-z_0)^n = f(z_0) + f'(z_0)\cdot(z-z_0) + \frac{f''(z_0)}{2!}(z-z_0)^2 + \dots$$

Theorem 1: If f is analytic in $D_R(z_0) = \{z \in \mathbb{C} \mid |z-z_0| < R\}$, then

a) Taylor series $\longrightarrow f(z)$ $\forall z \in D_R(z_0)$

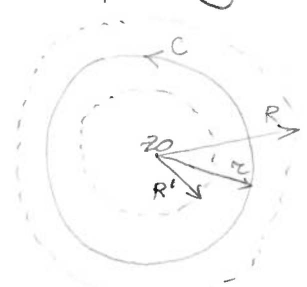
b) Taylor series $\xrightarrow{\text{uniformly}} f(z)$ on $D_{R'}(z_0)$ for any $0 < R' < R$

► b) $\Rightarrow$ a): For any $z \in D_R(z_0)$, choose $R' = \frac{|z| + R}{2}$ and apply b)

To prove b), we shall apply Cauchy integral formula $f(z) = \frac{1}{2\pi i}\int_C \frac{f(w)\,dw}{w-z}$ where C is a circle of radius r centered at z_0, with $R' < r < R$.

For simplicity, set $r = \frac{R'+R}{2}$. In above integration: $w \in C$, i.e. $|w - z_0| = r$

$z \in D_{R'}(z_0)$, i.e. $|z - z_0| < R'$



$$\left.\begin{array}{l} \dfrac{1}{w-z} = \dfrac{1}{(w-z_0)-(z-z_0)} = \dfrac{1}{w-z_0}\cdot\dfrac{1}{1-\frac{z-z_0}{w-z_0}} \\ \underline{\text{But}}: \left|\dfrac{z-z_0}{w-z_0}\right| < \dfrac{R'}{r} < 1 \end{array}\right\} \Rightarrow$$

$$\Rightarrow \frac{1}{w-z} = \frac{1}{w-z_0}\left(1 + \frac{z-z_0}{w-z_0} + \left(\frac{z-z_0}{w-z_0}\right)^2 + \dots\right)$$

$$\underset{\forall n \geq 1}{=} \frac{1}{w-z_0}\left(1 + \frac{z-z_0}{w-z_0} + \left(\frac{z-z_0}{w-z_0}\right)^2 + \dots + \left(\frac{z-z_0}{w-z_0}\right)^n + \frac{\left(\frac{z-z_0}{w-z_0}\right)^{n+1}}{1-\frac{z-z_0}{w-z_0}}\right)$$

So:

$$f(z) = \underbrace{\underbrace{\frac{1}{2\pi i}\int_C \frac{f(w)}{w-z_0}dw}_{=f(z_0)} + \underbrace{\frac{z-z_0}{2\pi i}\int_C \frac{f(w)}{(w-z_0)^2}dw}_{\frac{f'(z_0)}{1!}(z-z_0)} + \dots + \underbrace{\frac{(z-z_0)^n}{2\pi i}\int_C \frac{f(w)}{(w-z_0)^{n+1}}dw}_{(z-z_0)^n\cdot\frac{f^{(n)}(z_0)}{n!}}}_{n^{\text{th}}\text{ Partial sum of the Taylor series!}} + \underbrace{\mathcal{E}_n(z)}_{\text{correction term}}$$

It remains to show $\mathcal{E}_n(z) \xrightarrow{\text{uniformly}} 0$ on $D_{R'}(z_0)$

Here: $\mathcal{E}_n(z) = \frac{1}{2\pi i}\int_C \frac{f(w)}{w-z}\cdot\left(\frac{z-z_0}{w-z_0}\right)^{n+1}dw \Rightarrow |\mathcal{E}_n(z)| \leq \underbrace{\frac{1}{2\pi}\cdot\underbrace{\text{length}(C)}_{=2\pi r}\cdot M\cdot\frac{2}{R-R'}\cdot\left(\frac{2R'}{R+R'}\right)^{n+1}}_{\xrightarrow{n\to\infty} 0}$

where $M = \max_{w\in C}|f(w)|$ - exists as $|f|$ - continuous

$$\left|\frac{z-z_0}{w-z_0}\right| < \frac{R'}{r} = \frac{2R'}{R+R'}, \quad |w-z| > r - R' = \frac{R-R'}{2} \Rightarrow \frac{1}{|w-z|} < \frac{2}{R-R'}$$

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- Recall that a power series is a formal sum $\sum_{n=0}^{\infty} a_n(z-z_0)^n$. Last time we stated the following basic result (which shall be proved now)

> **Theorem 2:** There is $0 \le R \le +\infty$ such that (R – radius of convergence)
>
> a) power series converges for $|z-z_0|<R$
>
> b) power series diverges for $|z-z_0|>R$
>
> c) power series converges uniformly on $D_{R'}(z_0)$ for any $0<R'<R$

Before proving this result, let's first present a formula for R

Def: The upper limit of a sequence $\{c_n\}_{n=1}^{\infty}$ of real numbers, denoted by $\limsup c_n$, is the smallest real number L s.t. $\forall \varepsilon>0$ there are only finitely many terms $c_n > L+\varepsilon$

Conventions: a) if no such L exists (e.g. $c_n=n$) then we set $L=+\infty$

b) if every real L has such property (e.g. $c_n=-n$), then set $L=-\infty$

Examples: 1) if $\{c_n\} \xrightarrow[n\to\infty]{} L$, then clearly $\limsup c_n = L$

2) if $c_n=(-1)^n$ (which has no limit), then $\limsup c_n = 1$

> Hadamard Formula for radius of convergence of $\sum_{n\ge 0} a_n(z-z_0)^n$:
> $$R = \frac{1}{\limsup \sqrt[n]{|a_n|}}$$

Note: To compute R usually try to apply ratio test (i.e. $R=\lim_{n\to\infty}\left|\frac{a_n}{a_{n+1}}\right|$ if this limit exists), then if it fails – apply root test (i.e. $R = 1/\lim_{n\to\infty}\sqrt[n]{|a_n|}$), and if it fails – then apply Hadamard f-la

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Proof of Theorem 2

We shall prove Thm 2 with R explicitly given by Hadamard formula.
Note that c) $\Rightarrow$ a) as in Theorem 1.

- Verification of c): uniform convergence on $D_{R'}(z_0)$ with $R' < R$

  Pick any $R' < r < R$ (e.g. can take $r = \frac{R'+R}{2}$) $\Rightarrow \frac{1}{R} < \frac{1}{r} < \frac{1}{R'}$. Let $k := \frac{1}{r}$.

  As $\frac{1}{R} = \limsup \sqrt[n]{|a_n|}$ and $k > \frac{1}{R}$, we conclude $\sqrt[n]{|a_n|} < k$ for $n > N$ for some N.

  Then: $|a_n(z-z_0)^n| = \left(\sqrt[n]{|a_n|}\cdot|z-z_0|\right)^n < (kR')^n \quad \forall z \in D_{R'}(z_0)$. But $0 < kR' < 1$ and $\sum_{n=0}^{\infty}(kR')^n$ converges!

  So, by comparison test: $\sum_{n=0}^{\infty} a_n(z-z_0)^n$ converges uniformly on $D_{R'}(z_0)$.

- Verification of b): diverges for $|z-z_0| > R$

  Let $|z-z_0| > R$ and pick any $|z-z_0| > r > R$, so that $\frac{1}{R} > \frac{1}{r} > \frac{1}{|z-z_0|}$

  Again, let $k := \frac{1}{r}$. Now $k < \frac{1}{R} = \limsup \sqrt[n]{|a_n|}$, hence, there are infinitely many terms with $\sqrt[n]{|a_n|} > k$. For such n:

  $$|a_n(z-z_0)|^n = \left(\sqrt[n]{|a_n|}\cdot|z-z_0|\right)^n > (k|z-z_0|)^n > 1$$

  Hence, $\sum_{n\geq 0} a_n(z-z_0)^n$ can not converge as we got a subsequence of terms with modulus > 1

$\blacksquare$

Now that we finally established these basic results, we can address the question if every power series is a Taylor series first, and then if disk of convergence coincides with the Taylor disk (defined as max disk $D_R(z_0)$ s.t. f is analytic on it).

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Lemma 1: If power series $\sum_{n=0}^{\infty} a_n(z-z_0)^n$ converges at some z_1 with $r=|z_1-z_0|$ then it converges for all $z \in D_r(z_0)$, and moreover converges uniformly on $D_{r'}(z_0)$ for any $0<r'<r$

▶ Follows from Thm 2 (or can be directly derived from comparison test) ∎

Lemma 2: If $\{f_n\}_{n=0}^{\infty}$ – sequence of continuous functions on D – region of $\mathbb{C}$ and $f_n \xrightarrow{\text{uniformly}} f$, then f – continuous

▶ Same argument as in the calculus class!
For any $z_0 \in D$ and any $\varepsilon>0$, there is N s.t. $|f_N(z)-f(z)|<\frac{\varepsilon}{3}$ $\forall z \in D$.
As f_N – continuous, there is $\delta>0$ s.t. $|f_N(z)-f_N(z_0)|<\frac{\varepsilon}{3}$ $\forall z \in D_\delta(z_0)$

Then: $|f(z)-f(z_0)| \le |f(z)-f_N(z)| + |f_N(z)-f_N(z_0)| + |f_N(z_0)-f(z_0)| < 3\cdot\frac{\varepsilon}{3} = \varepsilon$
$\forall z \in D_\delta(z_0)$ ∎

Lemma 3: If $\{f_n(z)\}$ – sequence of continuous functions on set D in $\mathbb{C}$ containing contour Γ and $f_n \xrightarrow{\text{uniformly}} f$ on D, then

$$\int_\Gamma f_n(z)\,dz \xrightarrow[n\to\infty]{} \int_\Gamma f(z)\,dz$$

▶ Same as in calculus.
Indeed, $\forall \varepsilon>0$ $\exists N$ s.t. $|f_n(z)-f(z)|<\varepsilon$ $\forall n \ge N$ $\forall z \in D$

Hence: $\left|\int_\Gamma f_n(z)\,dz - \int_\Gamma f(z)\,dz\right| = \left|\int_\Gamma (f_n(z)-f(z))\,dz\right| \le \varepsilon \cdot \text{length}(\Gamma)$ ∎

Lemma 4: Let $\{f_n(z)\}$ – analytic f-s in simply-connected domain D, s.t. $f_n \xrightarrow{\text{uniformly}} f$ on D. Then f – analytic.

▶ By Lemma 2: f – continuous. By Lemma 3: $\int_\Gamma f(z)\,dz = 0$ $\forall$ closed curve $\Gamma \subset D$
But then f is analytic by Morera's theorem! ∎

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Theorem 3: A power series $\sum_{n=0}^{\infty} a_n (z-z_0)^n$ with a non-zero radius of convergence $R>0$ sums to a function $f(z)$ that is analytic in $D_R(z_0)$

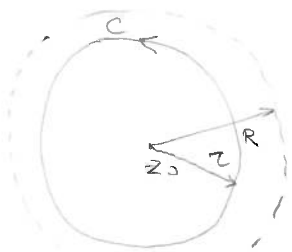
► For any $z \in D_R(z_0)$ consider $R' = \frac{R+|z-z_0|}{2}$ and disk $D_{R'}(z_0)$

Then $\sum_{j=0}^{n} a_j (z-z_0)^j$ are analytic (just polynomials) and converge uniformly on $D_{R'}(z_0)$ by Theorem 2 $\Rightarrow$ their limit is analytic by Lemma 4 ∎

Moreover, in the setup of Thm 3: the original power series coincides with the Taylor series of the limit $f(z)$ around z_0:

Theorem 4: If $\sum_{n=0}^{\infty} a_n (z-z_0)^n$ has R.O.C. $R>0$ and $f(z)$ is its analytic value inside $D_R(z_0)$, then $a_n = \frac{f^{(n)}(z_0)}{n!}$ for all n, so that the above power series is the Taylor series for f

►



$$f^{(n)}(z_0) = \frac{n!}{2\pi i} \int_C \frac{f(w)}{(w-z_0)^{n+1}}\, dw$$

But $\sum_{j=0}^{n} a_j (z-z_0)^j \xrightarrow{\text{uniformly}} f(z)$ on $D_r(z_0)$

$$\Rightarrow \sum_{j=0}^{n} \frac{a_j (z-z_0)^j}{(z-z_0)^{n+1}} \xrightarrow{\text{uniformly}} \frac{f(z)}{(z-z_0)^{n+1}}$$ on the annulus $\{w : r-\varepsilon < |w-z_0| < r+\varepsilon\}$ for small $\varepsilon>0$,

By Lemma 4: $$\int_C \frac{f(w)}{(w-z_0)^{n+1}}\, dw = \sum_{j=0}^{\infty} \underbrace{\int_C a_j (z-z_0)^{j-n-1}\, dz}_{= \begin{cases} a_n \cdot 2\pi i & \text{if } j=n \\ 0 & \text{if } j\neq n \end{cases}} = 2\pi i \cdot a_n$$

So: $f^{(n)}(z_0) = n! \cdot a_n \Rightarrow \boxed{a_n = \frac{f^{(n)}(z_0)}{n!}}$ ∎

Corollary: Convergence disk (interior of circle $D_R(z_0)$, $R = $ R.O.C) coincides with the Taylor disk (max disk $D_?(z_0)$ on which f - analytic)