Lecture #25

* Before we discuss key results today, let's establish two important facts:

Claim 1: If u is harmonic in a domain Ω and vanishes on some open disk $D_\varepsilon(z_0) \subseteq \Omega$, i.e. $u(z)=0\ \forall z \in D_\varepsilon(z_0)$, then $u(z)=0\ \forall z \in \Omega$

Note: In the last class we had a similar theorem for analytic functions, but the same argument will not apply to harmonic as zeros of harmonic functions are not isolated!

► The idea is standard by now: consider the function $g(z)=u_x - iu_y$ where $z=x+iy$. Then g satisfies CR on Ω as $(u_x)_x=(-u_y)_y$ b/c u - harmonic; $(u_x)_y=-(-u_y)_x$ - equality of mixed partials

Then $g(z)$ - analytic on Ω, $g\equiv 0$ on $D_\varepsilon(z_0)$, hence, by theorem from last time: $g(z)\equiv 0$ on $\Omega \Rightarrow u_x = 0 = u_y$ on $\Omega \Rightarrow u \equiv \text{const}$ on Ω

But $u\equiv 0$ on $D_\varepsilon(z_0) \Rightarrow u \equiv 0$ on Ω ∎

Claim 2: If $f(z)$ - analytic near z_0, then radius of convergence of its Taylor series equals radius of convergence of Taylor series of $f'(z)$ near z_0 (we shall denote this as $R_{f'}=R_f$).

► $R_{f'} \geq R_f$ as $f'(z)$ is analytic on $D_R(z_0)$ if $f(z)$ is analytic there.

Assume that $R_{f'} \neq R_f$, i.e. $R_{f'} > R_f$. Then, the Taylor series for $f'(z)$ would converge to some analytic function $g(z)$ on $D_{R_{f'}}(z_0)$

Then: antiderivative $G(z)$ of $g(z)$ is analytic on $D_{R_{f'}}(z_0)$ and

$$G'(z)=f'(z) \text{ on } D_{R_f}(z_0) \Rightarrow G(z)=f(z)+\underset{\text{constant}}{C} \text{ on } D_{R_f}(z_0)$$

But coefficients of Taylor series depend only on values in nbhd of z_0

$\Rightarrow G(z)$ & $f(z)$ have the same radius of convergence of Taylor series near $z_0 \Rightarrow$ contradiction $\Downarrow$. So $R_{f'}=R_f$ ∎

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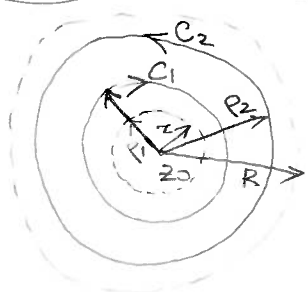
For the rest of today we shall be looking at washer = annulus

$$\Omega = \{ z \in \mathbb{C} \mid r < |z - z_0| < R \}$$

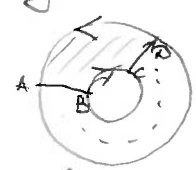
for some $z_0 \in \mathbb{C}$ and real $0 \leq r < R$

Thm 1: For any $r < \rho_1 < \rho_2 < R$, let $C_2 =$ positively oriented circle of radius ρ_2 around z_0 and $C_1 =$ negatively oriented circle of radius ρ_1 around z_0. For any $f(z)$ analytic on above washer/annulus, we have:

$$\int_{C_2} f(z)\,dz + \int_{C_1} f(z)\,dz = 0$$



▶ This is an immediate consequence of Cauchy theorem, as discussed a few weeks ago. Easy way to see it is to split into 2 domains and apply



$$\int_{AB} + \int_{BA} = 0 = \int_{C_2} + \int_{C_1}$$

Cauchy thm to each, and then add those ∎

In a completely similar fashion, the Cauchy integral formula implies the following result:

Thm 2: For any $\rho_1 < |a - z_0| < \rho_2$, we have

$$f(a) = \frac{1}{2\pi i} \int_{C_2} \frac{f(z)}{z-a}\,dz + \frac{1}{2\pi i} \int_{C_1} \frac{f(z)}{z-a}\,dz$$

We shall now use this result to represent $f(z)$, analytic on the above annulus $r < |z - z_0| < R$, by a power series <u>BUT</u> involving now both positive and negative powers of $z - z_0$

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Theorem 3 (main result today!): Let $f(z)$ be analytic in $r<|z-z_0|<R$. Then:

$$f(z)=\sum_{j=0}^{\infty} a_j(z-z_0)^j+\sum_{j=1}^{\infty} a_{-j}(z-z_0)^{-j} \text{ s.t.:}$$

a) both sums converge in above annulus

b) both sums converge uniformly in any subannulus
$\tilde{\rho}_1\le|z-z_0|\le\tilde{\rho}_2$ for any $r<\tilde{\rho}_1<\tilde{\rho}_2<R$

Moreover, $a_j=\frac{1}{2\pi i}\int_C \frac{f(w)}{(w-z_0)^{j+1}}\,dw$ for all $j\in\mathbb{Z}$ and any positively oriented simple closed curve C in annulus with z_0 inside it

[Note: This should be compared with Theorem 1 of Lecture 22, where we represented a function $f(z)$ analytic in $D_R(z_0)$ by its Taylor series

▶ Given any z in above $\Omega=\{z\in\mathbb{C}: r<|z-z_0|<R\}$, we can take $\tilde{\rho}_1=\frac{r+|z-z_0|}{2}$, $\tilde{\rho}_2=\frac{R+|z-z_0|}{2}$, so that b) $\Rightarrow$ a). Let's now prove b).

Given $r<\tilde{\rho}_1<\tilde{\rho}_2<R$, we shall pick ρ_1,ρ_2 s.t. $r<\rho_1<\tilde{\rho}_1<\tilde{\rho}_2<\rho_2<R$, e.g. similarly to Lecture 22, we shall take $\rho_1=\frac{r+\tilde{\rho}_1}{2}$, $\rho_2=\frac{R+\tilde{\rho}_2}{2}$, and let C_1, C_2 be the circles as in Thms 1&2. We shall now apply theorem 2 from above:

$$\boxed{f(z)=\frac{1}{2\pi i}\int_{C_2}\frac{f(w)}{w-z}\,dw+\frac{1}{2\pi i}\int_{C_1}\frac{f(w)}{w-z}\,dw}$$

Following word-for-word proof of Thm 1 of Lecture 22, based on

$$\frac{1}{w-z}=\frac{1}{(w-z_0)-(z-z_0)}=\frac{1}{w-z_0}+\frac{z-z_0}{(w-z_0)^2}+\frac{(z-z_0)^2}{(w-z_0)^3}+\dots,$$

we get

$$\frac{1}{2\pi i}\int_{C_2}\frac{f(w)}{w-z}\,dw=\sum_{j=0}^{n}a_j(z-z_0)^j+\varepsilon_n(z) \text{ with } \varepsilon_n(z)\xrightarrow[n\to\infty]{}0 \text{ uniformly in } \tilde{\rho}_1\le|z-z_0|\le\tilde{\rho}_2$$

$$a_j=\frac{1}{2\pi i}\int_{C_2}\frac{f(w)}{(w-z_0)^{j+1}}\,dw$$

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(Continuation)

To evaluate $\int_{C_1} \frac{f(w)}{w-z}\,dw$ we rather rewrite

$$\frac{1}{w-z} = \frac{1}{(w-z_0)-(z-z_0)} = -\frac{1}{z-z_0}\cdot\frac{1}{1-\frac{w-z_0}{z-z_0}} = -\frac{1}{z-z_0}\sum_{j=0}^{\infty}\left(\frac{w-z_0}{z-z_0}\right)^j \quad \text{as } \left|\frac{w-z_0}{z-z_0}\right|<1 \text{ for } w\in C_1$$

Fix any $n\geq 0$ and look at partial sum above, so that

$$\frac{1}{w-z} = -\frac{1}{z-z_0} - \frac{w-z_0}{(z-z_0)^2} - \dots - \frac{(w-z_0)^n}{(z-z_0)^{n+1}} + \frac{1}{w-z}\cdot\left(\frac{w-z_0}{z-z_0}\right)^{n+1}$$

So: $$\frac{1}{2\pi i}\int_{C_1}\frac{f(w)}{w-z}\,dw = a_{-1}\cdot(z-z_0)^{-1} + a_{-2}(z-z_0)^{-2} + \_\_ + a_{-n-1}(z-z_0)^{-n-1} + \tilde{\mathcal{E}}_n(z),$$

where $$a_{-j} = -\frac{1}{2\pi i}\int_{C_1} f(w)(w-z_0)^{j-1}\,dw \text{ for } j\geq 1$$
$$= \frac{1}{2\pi i}\int_{-C_1}\frac{f(w)}{(w-z_0)^{-j+1}}\,dw$$
$\uparrow$ same circle but now positively oriented

and
$$\tilde{\mathcal{E}}_n(z) = \frac{1}{2\pi i}\int_{C_1}\frac{f(w)}{w-z}\left(\frac{w-z_0}{z-z_0}\right)^{n+1}dw.$$

But we can estimate this integral as before:

$$\left.\begin{array}{l} |f(w)|\leq M \ \forall w\in C_1 \\ |w-z|\geq \tilde{\rho}_1-\rho_1 \Rightarrow \frac{1}{|w-z|}\leq \frac{1}{\tilde{\rho}_1-\rho_1} \\ \left|\frac{w-z_0}{z-z_0}\right| \leq \frac{\rho_1}{\tilde{\rho}_1} < 1 \end{array}\right\} \Rightarrow |\tilde{\mathcal{E}}_n(z)| \leq \frac{1}{2\pi}\cdot 2\pi\rho_1\cdot M\cdot\frac{1}{\tilde{\rho}_1-\rho_1}\left(\rho_1/\tilde{\rho}_1\right)^n$$

$\Downarrow$

$\tilde{\mathcal{E}}_n(z) \xrightarrow[n\to\infty]{} 0$ uniformly on the annulus $\tilde{\rho}_1\leq|z-z_0|\leq\tilde{\rho}_2$.

The rest of the theorem follows now $\blacksquare$

[Note: Unlike in Lecture 22, we are no longer saying that a_j equals $\frac{1}{j!}f^{(j)}(z_0)$ for $j\geq 0$ (as the latter may be not defined even)

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Def: The above sum $\sum_{j=-\infty}^{+\infty} a_j (z-z_0)^j$ is called Laurent series of $f(z)$

We also have the following analogue of Theorems 3-4 from Lecture 22:

Theorem 4: If $\sum_{j=0}^{+\infty} c_j (z-z_0)^j$ converges inside $C_R(z_0)$

$\sum_{j=1}^{\infty} c_j (z-z_0)^{-j}$ converges outside $C_r(z_0)$

and $r<R$, then there is an analytic function $f(z)$ in the washer $r<|z-z_0|<R$ whose Laurent series in it is exactly $\sum_{j=0}^{+\infty} c_j (z-z_0)^j + \sum_{j=1}^{\infty} c_{-j} (z-z_0)^{-j}$.

We shall skip the proof for now (as it's also omitted in the book)

Note. If $r=0$, then the annulus above is just $D_R(z_0)\setminus\{z_0\}$ = punctured open disk.

Def: z_0 - isolated singularity of $f(z)$ if $f(z)$ is analytic on $D_\varepsilon(z_0)\setminus\{z_0\}$ (for some $\varepsilon>0$) but is not analytic at z_0 itself

Looking at the Laurent series of $f(z)$ in $0<|z-z_0|<R$, we shall distinguish 3 cases:

Def: a) If $a_{<0}=0$, then z_0 is called removable singularity.

b) If $a_{-m}\neq 0$ & $a_{<-m}=0$ for some $m>0$, then z_0 is called a pole of order m

c) If $a_{-j}\neq 0$ for ∞ many $j>0$, then z_0 - essential singularity

Examples: $\frac{z^2-1}{z-1}$ has $z_0=1$ as removable singularity

$\frac{\sin z}{z}$ has $z_0=0$ as removable singularity

$\frac{\sin z}{z^3}$ has $z_0=0$ as a pole of order 2

$e^{1/z}$ has $z_0=0$ as essential singularity