Lecture #26

* Last time:

1) Any analytic f-n $f(z)$ on the annulus $r<|z-z_0|<R$ has a Laurent series
$$f(z)=\sum_{j=0}^{\infty} a_j(z-z_0)^j+\sum_{j=1}^{\infty} a_{-j}(z-z_0)^{-j}$$
with both sums converging in this annulus (converging uniformly in any subannulus $\rho_1\le|z-z_0|\le\rho_2$ $\forall\, r<\rho_1<\rho_2<R$)

[Rmk: While we had explicit formulas for $\{a_j\}_{j=-\infty}^{+\infty}$ in terms of integrals, in practice they are rarely used (as we shall see)

2) Applying the above to $r=0$, we obtain three possibilities for an isolated singularity of $f(z)$ at z_0:

a) z_0 - removable singularity if $a_{-j}=0$ $\forall j>0$.

b) z_0 - pole of order $m>0$ if $a_{-m}\neq 0$, $a_{-j}=0$ $\forall j>m$

c) z_0 - essential singularity if there are ∞ many $j>0$ s.t. $a_{-j}\neq 0$.

Examples:

1) $f(z)=\frac{z^2}{z}$ has a removable singularity at $z_0=0$, where it's not defined. Indeed, $f(z)=z$ on $\mathbb{C}\setminus\{0\}$ $\Rightarrow$ its Laurent series has $a_j=\begin{cases}1, & j=1\\ 0, & j\neq 1\end{cases}$.

2) $f(z)=\frac{\sin z}{z}$ has a removable singularity at $z_0=0$, where it's not defined. Indeed, $\sin z$ has a Taylor series $z-\frac{z^3}{3!}+\frac{z^5}{5!}-\dots$ whose radius of convergence is $R=\infty \Rightarrow \sin z=z-\frac{z^3}{3!}+\frac{z^5}{5!}-\dots \Rightarrow \frac{\sin z}{z}=1-\frac{z^2}{3!}+\frac{z^4}{5!}-\dots$ for any $z\neq 0$ and the latter sum converges everywhere. Hence, Laurent series of $f(z)$ at $z_0=0$ is $1-\frac{z^2}{3!}+\frac{z^4}{5!}-\dots \Rightarrow a_{-j}=0$ $\forall j>0$

3) $f(z)=\frac{\sin z}{z^4}$ has a pole of order 3 at $z_0=0$ (arguing as above we get Laurent series $\frac{1}{z^3}-\frac{1}{3!}\cdot\frac{1}{z}+\frac{1}{5!}z-\dots$)

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Ex 1: Describe zeroes & singularities of $f(z)=\cos(\frac{1}{z})$.

Zeroes

Recall that $\cos w = 0 \iff w = \frac{\pi}{2}+\pi k,\ k\in\mathbb{Z}$.

Hence, zeroes of $f(z)$ are $z=\frac{1}{\frac{\pi}{2}+\pi k} = \frac{2}{2k+1}\cdot\frac{1}{\pi},\ k\in\mathbb{Z}$

We also have to describe order of each zero!

As $f'(z) = -\sin(\frac{1}{z})\cdot\frac{-1}{z^2} = \frac{\sin(1/z)}{z^2}$ and $\sin(\frac{1}{z})\neq 0$ for above $z \Rightarrow$ all of order 1

So: $f(z)$ has order 1 zeroes at $\{\frac{2}{(2k+1)\pi} \mid k \text{ - integer}\}$

Poles

Recall that $\cos w = 1-\frac{w^2}{2!}+\frac{w^4}{4!}-\frac{w^6}{6!}+\ldots$ has $R=\infty$ (radius of converg.)

$\Rightarrow \cos\frac{1}{z} = 1-\frac{1}{2!}\cdot z^{-2}+\frac{1}{4!}z^{-4}-\frac{1}{6}z^{-6}+\ldots$

Hence, the above is a Laurent series of $f(z)$ near the only isolated singularity $z_0=0$ (where f is not defined) $\Rightarrow$ $z_0=0$ - essential singularity

Let's now discuss several structural results that allow to understand which of 3 types singularity we have by analyzing $f(z)$ as $z\to z_0$

Lemma 1: If $f(z)$ has a removable singularity at z_0, then:

i) f - bounded in $D_\varepsilon(z_0)\setminus z_0$ for some $\varepsilon>0$

ii) f has a finite limit as $z\to z_0$

iii) $g(z)=\begin{cases} f(z), & z\in D_\varepsilon(z)\setminus z_0 \\ \lim_{\tilde z\to z_0} f(\tilde z), & z=z_0\end{cases}$ is analytic on $D_\varepsilon(z_0)$

Indeed, by definition, Laurent series of $f(z)$ near z_0 is $\sum_{j=0}^{\infty} a_j(z-z_0)^j$ which converges on $D_\varepsilon(z_0)\setminus z_0$, but also at z_0 where it equals a_0.

Hence, by Theorem 3 of Lecture 22, this Laurent = power series sums to a function $g(z)$ analytic on $D_\varepsilon(z_0)$. All parts follow now

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As above lemma shows, removable singularities are not really of big interest.

Lemma 2: If $f(z)$ has a pole of order $m>0$ at z_0, then:

i) $(z-z_0)^m f(z)$ has a removable singularity at z_0

ii) $|(z-z_0)^\ell f(z)| \xrightarrow[z\to z_0]{} \infty$ $\forall$ $0\le \ell<m$

iii) $f(z)\to\infty$ as $z\to z_0$

▶ As $f(z)=\frac{a_{-m}}{(z-z_0)^m}+\frac{a_{1-m}}{(z-z_0)^{m-1}}+\dots$ converging in some $D_\varepsilon(z_0)\setminus z_0$, we get

$(z-z_0)^m f(z) = a_{-m}+a_{1-m}(z-z_0)+\dots$ converging in the same $D_\varepsilon(z_0)\setminus z_0$,

but also at $z=z_0$ (as we get $a_{-m}+0+0+\dots$)

Hence, as in the proof of Lemma 1, we see that $(z-z_0)^m f(z)$ has a removable singularity and $(z-z_0)^m f(z) \xrightarrow[z\to z_0]{} a_{-m}\neq 0$. Hence $\forall\, 0\le \ell<m$:

$|(z-z_0)^\ell f(z)| = \left|\frac{(z-z_0)^m f(z)}{(z-z_0)^{m-\ell}}\right| \xrightarrow[z\to z_0]{} \infty$. For $\ell=0$, this implies $f(z)\xrightarrow[z\to z_0]{}\infty$ ■

Def: z_0 is called a simple pole of $f(z)$ if it is a pole of order 1

Corollary 1: $f(z)$ has a pole of order m at z_0 $\iff$ in some $D_\varepsilon(z_0)\setminus z_0$:

$f(z)=\frac{g(z)}{(z-z_0)^m}$ with $g(z)$ analytic in $D_\varepsilon(z_0)$ and $g(z_0)\neq 0$.

▶ $\Rightarrow$: follows from above proof of Lemma 2

$\Leftarrow$: if $g(z)$ - analytic in $D_\varepsilon(z_0)$ & $g(z_0)\neq 0$ $\Rightarrow$ has Taylor series

$g(z)=a_0+a_1(z-z_0)+a_2(z-z_0)^2+\dots$ converging on $D_\varepsilon(z_0)$

$\Rightarrow \frac{g(z)}{(z-z_0)^m}=\frac{a_0}{(z-z_0)^m}+\frac{a_1}{(z-z_0)^{m-1}}+\dots$ converges on $0<|z-z_0|<\varepsilon$

$\Rightarrow$ it gives rise to analytic function $f(z)$ on $0<|z-z_0|<\varepsilon$ with above Laurent series, which has pole of order m at z_0 ■

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Theorem (Picard's thm): A function $f(z)$ with essential singularity at z_0 assumes every complex number, with possibly one exception, as a value in any punctured disk $D_\varepsilon(z_0)\setminus\{z_0\}$

Note: In other words for any $w \in \mathbb{C}$, except for possibly one value, and for any $\varepsilon>0$ there are actually ∞ many points $z \in D_\varepsilon(z_0)\setminus\{z_0\}$ with $f(z)=w$.

Combining above Lemma 1, Lemma 2, Theorem, we arrive at:

Upshot: One can determine type of singularity of $f(z)$ at z_0 by looking at the values of $f(z)$ as $z \to z_0$. Explicitly:

1) z_0 - removable singularity $\Leftrightarrow$ $f(z)$ has a finite limit as $z \to z_0$
2) z_0 - pole $\Leftrightarrow$ $f(z) \to \infty$ as $z \to z_0$.
3) z_0 - essential singularity $\Leftrightarrow$ $f(z)$ has no finite/infinite limit $\Leftrightarrow$ $f(z)$ is neither bounded near z_0 nor $\to \infty$ as $z \to z_0$

Remark: a) Looking back at Ex 1, we could argue that $z_0=0$ is essential singularity by noting that $\forall \varepsilon>0$ $\cos\frac{1}{z}$ varies all the way from -1 to 1 as $z \in (-\varepsilon,\varepsilon) \subseteq \mathbb{R} \subseteq \mathbb{C}$ and has no limit as $z \to 0$ there

b) Note that when viewing $\cos\frac{1}{z}$ on $\mathbb{R}$ it is bounded, but once we view it on $\mathbb{C}$, then we have Picard's Theorem!

Ex 2: Classify zeroes and singularities of $f(z)=\frac{\tan z}{z}$

* Home Extra Reading: Read "Laurent expansion" worksheet (posted in Brightspace)