Lecture #27

- Let's start from the following problem (50% of old quiz in MA425):

Ex1: For each of the following function $f(z)$ & point z_0, determine the type of singularity of f at z_0:

a) removable

b) pole

c) essential

d) not an isolated singularity

1) $f(z) = \frac{1-\cos z}{z^2}$, $z_0 = 0$

2) $f(z) = \text{Log} z$, $z_0 = 0$

3) $f(z) = \cot z - \frac{1}{z}$, $z_0 = 0$

4) $f(z) = \frac{z}{e^z + 1}$, $z_0 = \pi i$

1) $\cos z = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \dots \Rightarrow 1 - \cos z = \frac{z^2}{2!} - \frac{z^4}{4!} + \dots \Rightarrow f(z) = \frac{1}{2!} - \frac{z^2}{4!} + \dots \quad \forall z \neq 0$

(as radius of convergence $= \infty$)

We note that above series converges wherever Taylor series of $\cos z$ does

So: a) removable singularity

2) $\text{Log} z$ is not even continuous on any $D_\varepsilon(0) \setminus \{0\}$ for $\varepsilon > 0$

So: d) not isolated singularity.

3) $\left.\begin{array}{l} \cos z = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \dots \\ \sin z = z - \frac{z^3}{3!} + \dots \end{array}\right\} \Rightarrow \cot z = \frac{1 - \frac{z^2}{2!} + \dots}{z - \frac{z^3}{3!} + \dots} = \frac{1 - \frac{z^2}{2!} + \dots}{z\left(1 - \frac{z^2}{3!} + \dots\right)}$

$= \frac{1}{z} \cdot \left(1 - \frac{z^2}{2!} + \dots\right)\left(1 + \frac{z^2}{3!} + \dots\right) = \frac{1}{z} - \frac{1}{3} z + \dots$

$\Rightarrow f(z) = -\frac{1}{3} z + (\dots)$ higher order terms

So: a) removable singularity

Lecture #27

(Continuation)

4) e^z around $z_0 = \pi i$ has Taylor series $\sum_{n\geq 0} \frac{(e^z)^{(n)}}{n!}\Big|_{z=z_0} \cdot (z-z_0)^n$, which is $-1-(z-\pi i)-\frac{1}{2!}(z-\pi i)^2-\ldots \Rightarrow e^z+1 = -(z-\pi i)-\frac{1}{2!}(z-\pi i)^2-\ldots$

Numerator z has Taylor series $\pi i + (z-\pi i)$.

Thus: $f(z) = \dfrac{\pi i + (z-\pi i)}{-(z-\pi i)-\frac{1}{2!}(z-\pi i)^2-\ldots} = -\dfrac{1}{z-\pi i}\left(\pi i + (z-\pi i)\right)\left(1-\frac{1}{2}(z-\pi i)+\ldots\right)$

which clearly has a pole of order 1.

So: b) pole.

Def: Poles of order 1 are usually called simple poles.

A simple observation relates poles/zeroes of f to zeroes/poles of $\frac{1}{f}$:

Lemma 1: a) If f has a zero of order m at z_0, then $\frac{1}{f(z)}$ has a pole of order m at z_0.

b) If f has a pole of order m at z_0, then $\frac{1}{f(z)}$ has a removable singularity at z_0 and if we consider $g(z) = \begin{cases} 1/f(z), & z \neq z_0 \\ 0, & z = z_0 \end{cases}$ then it has a zero of order m at z_0.

▸ a) $f(z) = a_m(z-z_0)^m + a_{m+1}(z-z_0)^{m+1} + \ldots$ – Taylor series expansion

$$\Downarrow \quad \frac{1}{f(z)} = \frac{1}{a_m(z-z_0)^m} \cdot \frac{1}{1+\frac{a_{m+1}}{a_m}(z-z_0)+\frac{a_{m+2}}{a_m}(z-z_0)^2+\ldots}$$

Note: Denominator converges in small nbhd $D_\varepsilon(z_0)$ where $f(z)$ is analytic, and as $z \to z_0$ it tends to 1, hence $\left|\frac{a_{m+1}}{a_m}(z-z_0)+\frac{a_{m+2}}{a_m}(z-z_0)^2+\ldots\right| < 1$ for $|z-z_0|<\varepsilon$ (some $\varepsilon>0$) and we can apply geometric progression to conclude

$$\frac{1}{f(z)} = \frac{1}{a_m(z-z_0)^m}\left(1-\frac{a_{m+1}}{a_m}(z-z_0)+\ldots\right) \Rightarrow \text{has order } m \text{ pole at } z_0$$

Lecture #27

(Continuation)

b) If $f(z) = \frac{a_{-m}}{(z-z_0)^m} + \frac{a_{1-m}}{(z-z_0)^{m-1}} + \dots$ with $a_{-m} \neq 0$, then

writing $f(z) = \frac{a_{-m}}{(z-z_0)^m}\left(1 + \frac{a_{1-m}}{a_{-m}}(z-z_0) + \dots\right)$ we again use geometric progression series to conclude

$$\frac{1}{f(z)} = \frac{(z-z_0)^m}{a_{-m}}\left(1 - \frac{a_{1-m}}{a_{-m}}(z-z_0) + \dots\right) \quad \text{for } z \in D_\varepsilon(z_0)\backslash z_0$$

ε - small enough so that f - analytic there

As $f(z_0)$ is not well-defined, so is $\frac{1}{f}(z_0)$, but the above implies that $g(z) := \begin{cases} 1/f(z), & \forall z \in D_\varepsilon(z_0)\backslash z_0 \\ 0, & z = z_0 \end{cases}$ is analytic on $D_\varepsilon(z_0)$ and has order m zero at z_0 ∎

Ex2: Classify zeroes & singularities of $\frac{\sin z}{(z^2-\pi^2)^3}$.

- Zeroes: possible at $z = \pi k$, $k \in \mathbb{Z}$
- Poles: possible at $z = \pm\pi$.

$\Rightarrow$ $\boxed{\text{At } z = \pi k \text{ with } k \in \mathbb{Z}\backslash\{\pm1\} \text{ have zeroes and } \cos(\pi k) \neq 0 \to \text{order } 1}$

Warning: As both $z_0 = \pi$ and $z_0 = -\pi$ appear in both lists, we need to take a closer look there!

Case $z_0 = \pi$

Taylor series of $\sin z$ around z_0 is $\sin z_0 + \frac{\sin' z_0}{1!}(z - z_0) + \dots$

For $z_0 = \pi$: $\cos \pi = -1 \Rightarrow \sin z = -(z-\pi) + \dots$ ← higher powers of $z-\pi$

Taylor series of $(z^2-\pi^2)^3$ around $z_0 = \pi$ is $(z-\pi)^3 \times \underbrace{\begin{pmatrix}\text{Taylor series} \\ \text{of } (z+\pi)^3 \text{ at } z_0\end{pmatrix}}_{8\pi^3 + *(z-\pi) + \dots}$

$\Rightarrow$ denominator $= 8\pi^3(z-\pi)^3 + \dots$ ← higher powers of $z-\pi$

So: $f(z) = \frac{\sin z}{(z^2-\pi^2)^3}$ which is analytic in $D_\varepsilon(\pi)\backslash\{\pi\}$ has Laurent series

$$\frac{-(z-\pi) + \dots}{8\pi^3(z-\pi)^3(1 + \dots)} = -\frac{1}{8\pi^3}(z-\pi)^{-2} + \dots$$ ← higher powers

$\Rightarrow \boxed{z_0 = \pi - \text{pole of order } 2!}$

Case $z_0 = -\pi$: Similar analysis $\Rightarrow \boxed{-\pi - \text{pole of order } 2!}$

Lecture #27

Ex 3: Classify poles of $f(z) = \frac{\tan z}{z}$

▶ $f(z) = \frac{\sin z}{\cos z \cdot z}$

Possible poles: $z = 0$ & $z = \frac{\pi}{2} + \pi k$, $k \in \mathbb{Z}$.

- $z_0 = 0$

Last time $\frac{\sin z}{z} = 1 - \frac{z^2}{3!} + \dots$ has removable singularity at $z_0 = 0$

Also $\frac{1}{\cos z} = \frac{1}{1 - \frac{z^2}{2!} + \dots} = 1 + \frac{z^2}{2!} + \dots$ is analytic in nbhd of z_0

So: $z_0 = 0$ is not a pole!

- $z_0 = \frac{\pi}{2} + \pi k$, $k \in \mathbb{Z}$

$\frac{\sin z}{z}$ - analytic in nbhd of z_0, not vanishing at z_0 $\Rightarrow$ $\frac{\sin z}{z} = \underset{\neq 0}{b_0} + b_1(z - z_0) + \dots$

$\cos z = \cos z_0 + \frac{\cos' z_0}{1!}(z - z_0) + \dots = 0 \pm (z - z_0) + \dots \Rightarrow \frac{1}{\cos z} = \underbrace{\pm \frac{1}{z - z_0}(1 + \dots)}_{\text{has pole of order } 1}$

So: All $\frac{\pi}{2} + \pi k$ ($k \in \mathbb{Z}$) - simple poles! ▯

Ex 4: Find Laurent expansion of $\frac{z^2 - z + 2}{z - 2}$ in $D = \{z \in \mathbb{C} \mid |z - 1| > 1\}$

▶ Note that $f(z) = \frac{z^2 - z + 2}{z - 2}$ is analytic in D. To compute Laurent series, we first compute Taylor series of $z^2 - z + 2$ around $z_0 = 1$:

$z^2 - z + 2 = 2 + (z - 1) + (z - 1)^2$ ($\leftarrow$ do you remember how to get this?)

On the other hand as $|z - 1| > 1 \Rightarrow \left|\frac{1}{z - 1}\right| < 1$ and so we re-expand:

$$\frac{1}{z - 2} = \frac{1}{(z - 1) - 1} = \frac{1}{(z - 1)(1 - \frac{1}{z - 1})} = \frac{1}{z - 1}\left(1 + \frac{1}{z - 1} + \left(\frac{1}{z - 1}\right)^2 + \dots\right) = \sum_{n=1}^{\infty} \frac{1}{(z - 1)^n}$$

Thus: $f(z) = \left[(z - 1)^2 + (z - 1) + 2\right] \cdot \left[\frac{1}{z - 1} + \frac{1}{(z - 1)^2} + \frac{1}{(z - 1)^3} + \dots\right]$

$$= (z - 1) + 2 + \sum_{n=1}^{\infty} \frac{4}{(z - 1)^n}$$

▯

[Note: For students who do not easily understand the above proof — come to office hours!

Lecture #27

Ex 5: For $f(z)=\frac{1}{(z-1)(z-2)}$ find its Laurent series in:

a) $|z|<1$

b) $1<|z|<2$

c) $|z|>2$

► Step 1 (partial fraction decomposition): $\frac{1}{(z-1)(z-2)}=\frac{1}{1-z}-\frac{1}{2-z}$.

a) if $|z|<1$, then by geometric progression:

$$\frac{1}{1-z}=1+z+z^2+\ldots$$

but also $\left|\frac{z}{2}\right|<1 \Rightarrow \frac{1}{2-z}=\frac{1}{2}\cdot\frac{1}{1-\frac{z}{2}}=\frac{1}{2}\left(1+\frac{z}{2}+\frac{z^2}{4}+\ldots\right)$

So: $\boxed{f(z)=(1+z+z^2+\ldots)-\left(\frac{1}{2}+\frac{1}{4}z+\frac{1}{8}z^2+\ldots\right)=\sum_{n=0}^{\infty}\left(1-\frac{1}{2^{n+1}}\right)z^n}$

b) if $1<|z|<2$, then we still have $\frac{1}{2-z}=\frac{1}{2}+\frac{1}{4}z+\frac{1}{8}z^2+\ldots$

but $\frac{1}{1-z}\neq 1+z+z^2+\ldots$ as the latter sum diverges!

Instead, we write $\frac{1}{1-z}=\frac{1}{-z\left(1-\frac{1}{z}\right)}=-\frac{1}{z}\left(1+\frac{1}{z}+\frac{1}{z^2}+\ldots\right)=-\frac{1}{z}-\frac{1}{z^2}-\frac{1}{z^3}-\ldots$

So: $\boxed{f(z)=\ldots-\frac{1}{z^3}-\frac{1}{z^2}-\frac{1}{z}-\frac{1}{2}-\frac{1}{4}z-\frac{1}{8}z^2-\ldots}$

c) if $|z|>2$, then we use $\frac{1}{1-z}=-\frac{1}{z}-\frac{1}{z^2}-\frac{1}{z^3}-\ldots$ as in b), but now have to change $\frac{1}{2-z}$ (as $\left|\frac{z}{2}\right|>1$ and so geom. progr. diverges)

To do so, we write

$$\frac{1}{2-z}=\frac{1}{-z\left(1-\frac{2}{z}\right)}\underset{\left|\frac{2}{z}\right|<1 \text{ now!}}{=}-\frac{1}{z}\left(1+\frac{2}{z}+\frac{4}{z^2}+\frac{8}{z^3}+\ldots\right)=-\frac{1}{z}-\frac{2}{z^2}-\frac{4}{z^3}-\ldots$$

So: $\boxed{f(z)=\left(-\frac{1}{z}-\frac{1}{z^2}-\frac{1}{z^3}-\ldots\right)-\left(-\frac{1}{z}-\frac{2}{z^2}-\frac{4}{z^3}-\ldots\right)=\sum_{n\geq 2}\frac{2^{n-1}-1}{z^n}}$

Moral: Two ways to write $\frac{1}{\lambda-\mu}=\frac{1}{\lambda}\cdot\frac{1}{1-\mu/\lambda}$ or $\frac{1}{\lambda-\mu}=-\frac{1}{\mu}\cdot\frac{1}{1-\lambda/\mu}$ and we choose 1st if $|\mu/\lambda|<1$ and second if $|\lambda/\mu|<1$

Lecture #27

Def: A *residue* of $f(z)$ at its isolated singularity z_0 is

$$\mathrm{Res}_{z_0}(f) := a_{-1}$$

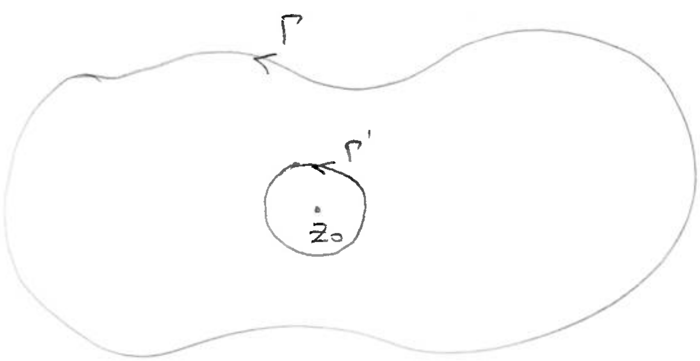
coefficient of $(z-z_0)^{-1}$ in Laurent series.

[Note: Your book also uses notation $\mathrm{Res}(f; z_0)$ or $\mathrm{Res}(z_0)$]

Cauchy theorem implies the following result:

Theorem: If Γ is a positively oriented simple closed curve, z_0 is inside Γ and $f(z)$ is analytic on and inside Γ besides point z_0, then:

$$\boxed{\int_\Gamma f(z)\,dz = 2\pi i \cdot \mathrm{Res}_{z_0}(f)}$$



By Cauchy theorem:

$$\int_\Gamma f(z)\,dz = \int_{\Gamma'} f(z)\,dz, \text{ where}$$

$\Gamma' = C_\varepsilon(z_0)$ – circle of small radius $\varepsilon > 0$ centered at z_0.

But $f(z) = \sum_{j=0}^{\infty} a_j (z-z_0)^j + \sum_{j=1}^{\infty} a_{-j}(z-z_0)^{-j}$ with both sums converging uniformly on Γ'

$$\Downarrow$$

$$\int_{\Gamma'} f(z)\,dz = \sum_{j=-\infty}^{+\infty} a_j \cdot \int_{\Gamma'} (z-z_0)^j\,dz = a_{-1} \cdot 2\pi i = \mathrm{Res}_{z_0}(f) \cdot 2\pi i$$

first calculated in Lecture 13!

∎