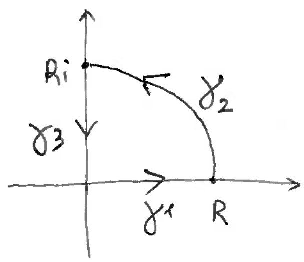
Lecture #30

Ex 1 (from old final): Evaluate $I = \int_0^{+\infty} \frac{x^2}{x^4+1}\,dx$ by integrating around quarter circle.

▶



$$\int_{\gamma_1} \frac{z^2}{z^4+1}\,dz + \int_{\gamma_2} \frac{z^2}{z^4+1}\,dz + \int_{\gamma_3} \frac{z^2}{z^4+1}\,dz = 2\pi i \cdot \operatorname{Res}_{\frac{1+i}{\sqrt{2}} = e^{i\cdot\pi/4}} \left(\frac{z^2}{z^4+1}\right)$$

Here: $\operatorname{Res}_{e^{i\cdot\frac{\pi}{4}}} \left(\frac{z^2}{z^4+1}\right) = \left.\frac{z^2}{4z^3}\right|_{z=e^{i\cdot\frac{\pi}{4}}} = \frac{1}{4}\cdot e^{i\cdot\frac{-\pi}{4}}$

Also: $\int_{\gamma_2} \frac{z^2}{z^4+1}\,dz \xrightarrow{R\to+\infty} 0$ as deg(denominator) = deg(numerator) + 2.

However, $\int_{\gamma_3} \frac{z^2}{z^4+1}\,dz$ can be also expressed as a multiple of I.

Parametrize $-\gamma_3$: it with $0 \le t \le R$, so that

$$\int_{-\gamma_3} \frac{z^2}{z^4+1}\,dz = \int_0^R \frac{(it)^2}{(it)^4+1}\cdot i\,dt = -i\int_0^R \frac{t^2}{t^4+1}\,dt \Rightarrow \int_{\gamma_3} \frac{z^2}{z^4+1}\,dz = i\cdot \int_{\gamma_1} \frac{z^2}{z^4+1}\,dz$$

Thus: $(1+i)I = \frac{2\pi i}{4} e^{i\cdot\frac{-\pi}{4}} \Rightarrow I = \frac{\pi i}{2\sqrt{2}} e^{i\cdot\frac{-\pi}{2}} = \boxed{\pi/2\sqrt{2}}$

▮

Today: § 6.4 = Improper integrals involving trigonometric functions

$$\int_{-\infty}^{+\infty} \frac{P(x)}{Q(x)} \cos(mx)\,dx \quad \& \quad \int_{-\infty}^{+\infty} \frac{P(x)}{Q(x)} \sin(mx)\,dx$$

with Q having no real roots.

Ex 2: p.v. $\int_{-\infty}^{+\infty} \frac{\cos(2x)}{1+x^2}\,dx = ?$

Note that if we naively write $\frac{\cos(2z)}{1+z^2} = \frac{e^{i\cdot 2z} + e^{i\cdot(-2z)}}{2(1+z^2)}$ and integrate either over the top half or the lower half of circle $|z| = R$ then the limit does not tend to 0 as $R \to +\infty$.

! However: $|e^{i\cdot 2z}| \le 1$ $\forall z \in \mathcal{H} = \{\operatorname{Im} > 0\}$, $|e^{i\cdot(-2z)}| \le 1$ $\forall z \in -\mathcal{H}$

which suggests to use both upper and lower halves accordingly.

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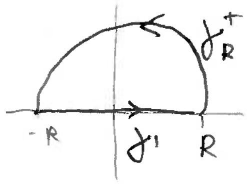
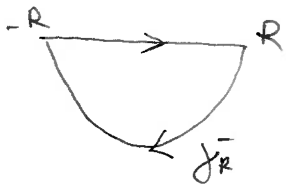
Solution of Ex 2

- 

$$\int_{\gamma'+\gamma_R^+} \frac{e^{i\cdot 2z}}{2(1+z^2)}dz = 2\pi i \cdot \mathrm{Res}_i\left(\frac{e^{i\cdot 2z}}{2(1+z^2)}\right) = 2\pi i \cdot \frac{e^{i\cdot 2z}}{4z}\Big|_{z=i} = \frac{\pi}{2}e^{-2}$$

$$\int_{\gamma_R^+} \frac{e^{i\cdot 2z}}{2(1+z^2)}dz \xrightarrow{R\to+\infty} 0$$

$$\Rightarrow \int_{\gamma'} \frac{e^{i\cdot 2z}}{2(1+z^2)}dz \to \frac{\pi}{2e^2} \quad \text{as } R\to+\infty$$

- 

(as contour is negatively oriented)

$$\int_{\gamma'+\gamma_R^-} \frac{e^{i\cdot(-2z)}}{2(1+z^2)}dz = -2\pi i\, \mathrm{Res}_{-i}\frac{e^{i\cdot(-2z)}}{2(1+z^2)} = -2\pi i \cdot \frac{e^{-2iz}}{4z}\Big|_{z=-i} = \frac{\pi}{2e^2}$$

$$\int_{\gamma_R^-} -\text{''}- \xrightarrow{R\to+\infty} 0$$

$$\Rightarrow \int_{\gamma'} \frac{e^{i\cdot(-2z)}}{2(1+z^2)}dz = \frac{\pi}{2e^2}.$$

Therefore: $\text{p.v.}\int_{-\infty}^{+\infty} \frac{\cos 2x}{1+x^2}dx = \frac{\pi}{2e^2} + \frac{\pi}{2e^2} = \boxed{\pi/e^2}$

Remark: The above exercise can actually be solved much faster as

$$\text{p.v.}\int_{-\infty}^{+\infty} \frac{\cos 2x}{1+x^2}dx = \mathrm{Re}\Big(\underbrace{\text{p.v.}\int_{-\infty}^{+\infty}\frac{e^{i\cdot 2x}}{1+x^2}dx}\Big) = \frac{\pi}{e^2}$$

can be computed as last time to be $2\pi i \cdot \mathrm{Res}_i\left(\frac{e^{i\cdot 2x}}{1+x^2}\right) = \frac{\pi}{e^2}$.

The above will always work when deg(denominator) ≥ deg(numerator) + 2.

Ex 3: $\text{p.v.}\int_{-\infty}^{+\infty} \frac{\cos(2x)}{x-3i}dx = ?$

! Note: now deg(denominator) = deg(numerator) + 1, so above doesn't apply

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Luckily, we can still assert $\int_{\gamma_R^+} \ldots \to 0$ as $R \to +\infty$ due to next result:

Theorem (Jordan's Lemma):

a) If $m>0$, $\frac{P(x)}{Q(x)}$ - ratio of polynomials with $\deg(Q) \geq \deg(P)+1$, then

$$\lim_{R\to+\infty} \int_{\gamma_R^+} e^{imz} \frac{P(z)}{Q(z)}\,dz = 0$$

b) If $m<0$, — " —, then

$$\lim_{R\to+\infty} \int_{\gamma_R^-} e^{imz} \frac{P(z)}{Q(z)}\,dz = 0$$

Let's first apply this result to solve Ex 3:

Solution of Ex 3

γ_1 - line segment from $-R$ to R

γ_R^+ - upper half circle, γ_R^- - lower half circle

$\cos(2z) = \frac{1}{2}\left(e^{i\cdot 2z} + e^{i\cdot(-2z)}\right)$

- $\int_{\gamma_1+\gamma_R^+} \frac{e^{i\cdot 2z}}{z-3i}\,dz = 2\pi i \cdot e^{i\cdot 2\cdot 3i} = \frac{2\pi i}{e^6}$

  $\int_{\gamma_R^+} \frac{e^{i\cdot 2z}}{z-3i}\,dz \xrightarrow{R\to+\infty} 0$ by Thm a) above

  $\Rightarrow \text{p.v.} \int_{-\infty}^{+\infty} \frac{e^{i\cdot 2z}}{z-3i}\,dz = \frac{2\pi i}{e^6}$

- $\int_{\gamma_1+\gamma_R^-} \frac{e^{i\cdot(-2z)}}{z-3i}\,dz = 0$ as $\frac{e^{i(-2z)}}{z-3i}$ is analytic in $-\mathcal{H}$

  $\int_{\gamma_R^-} \frac{e^{i(-2z)}}{z-3i}\,dz \xrightarrow{R\to+\infty} 0$ by Thm b) above

  $\Rightarrow \text{p.v.} \int_{-\infty}^{+\infty} \frac{e^{i(-2z)}}{z-3i}\,dz = 0$

Thus: $\text{p.v.} \int_{-\infty}^{+\infty} \frac{\cos(2x)}{x-3i}\,dx = \boxed{\frac{\pi i}{e^6}}$ ∎

Note: As denominator involves $3i$, there is no "shortcut" as illustrated in the Remark on p. 2.

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Proof of Jordan's Lemma

► We'll treat only a), as b) is similar.

Parametrize γ_R^+: Re^{it} with $0 \le t \le \pi$, so that

$$\int_{\gamma_R^+} e^{imz} \frac{P(z)}{Q(z)} dz = \int_0^{\pi} e^{i \cdot m \cdot Re^{it}} \cdot \frac{P(Re^{it})}{Q(Re^{it})} \cdot Rie^{it} dt$$

Let $P(z) = a_0 + a_1 z + \_\_ + a_n z^n$, $a_n \neq 0$
$Q(z) = b_0 + b_1 z + \_\_ + b_n z^n + \_\_ + b_m z^m$, $b_m \neq 0$ & $m \geq n+1$ $\Big\} \Rightarrow$

$$\Rightarrow \frac{P(z)}{Q(z)} = \frac{1}{z^{m-n}} \cdot \underbrace{\frac{a_n + a_{n-1} \cdot \frac{1}{z} + \_\_ + a_0 \cdot \frac{1}{z^n}}{b_m + b_{m-1} \frac{1}{z} + \_\_ + b_0 \cdot \frac{1}{z^m}}}_{\to \frac{a_n}{b_m} \text{ as } z \to \infty}$$

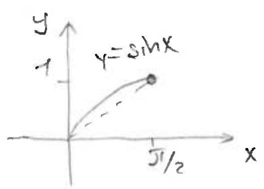
$$\Rightarrow \left| \frac{P(Re^{it})}{Q(Re^{it})} \right| \leq \frac{K}{R} \quad \text{for } K = \left|\frac{a_n}{b_m}\right| + 1 \in \mathbb{R}_{>0} \text{ and sufficiently large } R!$$

On the other hand:

$$|e^{imRe^{it}}| = e^{-mR \cdot \sin t}$$

So: $\left| \int_{\gamma_R^+} e^{imz} \frac{P(z)}{Q(z)} dz \right| \leq K \cdot \int_0^{\pi} e^{-mR\sin t} dt \overset{\text{symmetry}}{=\!=} 2K \cdot \underbrace{\int_0^{\pi/2} e^{-mR\sin t} dt}_{\text{remains to show } \to 0 \text{ as } R \to +\infty}$

But:



As $(\sin x)'' = -\sin x \leq 0$ $\forall\, 0 \le x \le \frac{\pi}{2} \Rightarrow \sin x \geq \frac{2}{\pi} x$

$$\Rightarrow -\sin t \leq -\frac{2}{\pi} t \Rightarrow \int_0^{\pi/2} e^{-mR\sin t} dt \leq \int_0^{\pi/2} e^{mR \cdot \frac{-2}{\pi} t} dt$$

However: $\displaystyle\int_0^{\pi/2} e^{-\frac{2mR}{\pi} t} dt = \frac{-\pi}{2mR} e^{-\frac{2mR}{\pi} t} \Big|_0^{\pi/2} = -\frac{\pi}{2mR}(e^{-mR} - 1) = \frac{\pi}{2mR}(1 - e^{-mR})$

As $\frac{\pi}{2mR} \xrightarrow{R \to +\infty} 0$, we get $\displaystyle\int_0^{\pi/2} e^{-mR\sin t} dt \to 0 \Rightarrow \int_{\gamma_R^+} e^{imz} \frac{P(z)}{Q(z)} dz \to 0$ □