Lecture #31

Today: § 6.5 (improper integrals with points of discontinuity)

Setup: Let $a<c<b$ and $f(x)$ be a function continuous on $[a,b]\setminus c$ but not continuous at c.

Then $\int_a^c f(x)dx = \lim_{\varepsilon\to 0^+} \int_a^{c-\varepsilon} f(x)dx$

$\int_c^b f(x)dx = \lim_{\varepsilon\to 0^+} \int_{c+\varepsilon}^b f(x)dx$

$\rightsquigarrow \int_a^b f(x)dx = \left(\int_a^c + \int_c^b\right) f(x)dx$ assuming both exist.

Example: a) Consider $f(x) = \frac{1}{\sqrt[3]{x}}$ on $[-1,1]$ with 0 being a value of c above

$$\int_\varepsilon^1 \frac{1}{\sqrt[3]{x}}dx = \frac{3}{2}x^{\frac{2}{3}}\Big|_\varepsilon^1 \xrightarrow{\varepsilon\to 0^+} 3/2$$

$$\int_{-1}^{-\varepsilon} \frac{1}{\sqrt[3]{x}}dx = \frac{3}{2}x^{\frac{2}{3}}\Big|_{-1}^{-\varepsilon} \xrightarrow{\varepsilon\to 0^+} -3/2$$

$$\Rightarrow \int_{-1}^{1} \frac{1}{\sqrt[3]{x}}dx = 0$$

b) Consider $f(x) = \frac{1}{x}$ on $[-1,1]$

$$\int_\varepsilon^1 \frac{1}{x}dx = \ln x\Big|_\varepsilon^1 = 1-\ln\varepsilon \xrightarrow{\varepsilon\to 0^+} \infty \Rightarrow \text{doesn't exist}$$

likewise $\int_{-1}^{-\varepsilon}\frac{1}{x}dx$, $\int_{-1}^{1}\frac{1}{x}dx$ don't exist.

To compensate for b) above, one defines principal value

$$\boxed{\text{p.v.}\int_a^b f(x)dx = \lim_{\varepsilon\to 0^+}\left(\int_a^{c-\varepsilon} + \int_{c+\varepsilon}^b\right)f(x)dx}$$

(when $\int_a^b f(x)dx$ exists, it coincides with p.v. $\int_a^b f(x)dx$).

Example: In part b) above: $\int_{-1}^{-\varepsilon}\frac{1}{x}dx + \int_\varepsilon^1 \frac{1}{x}dx = 0 \quad \forall\varepsilon \Rightarrow$

$$\Rightarrow \text{p.v.}\int_{-1}^1 \frac{1}{x}dx = 0$$

Finally, if $f(x)$ is continuous on $\mathbb{R}\setminus\{c\}$, then we define

$$\boxed{\text{p.v.}\int_{-\infty}^{+\infty} f(x)dx = \lim_{\substack{R\to+\infty\\ \varepsilon\to 0^+}}\left(\int_{-R}^{c-\varepsilon} + \int_{c+\varepsilon}^{R}\right)f(x)dx}$$

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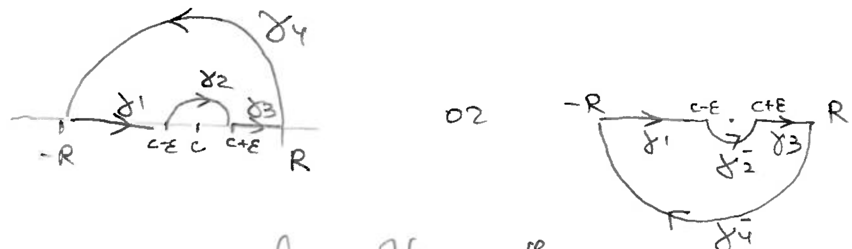
Following the strategy from the previous 2 lectures, we want to "close up" the segments $[-R, c-\varepsilon] \cup [c+\varepsilon, R]$ into a closed path not containing singularities on it.

Common choices:



[Note: One could also have one arch in $\mathcal{H}$, another in $-\mathcal{H}$ but this will only complicate computations as then one also needs to compute $Res_c(\dots)$

Finally, if $f(x)$ is continuous on $\mathbb{R} \setminus \{c_1, \dots, c_k\}$, then

$$\boxed{p.v. \int_{-\infty}^{+\infty} f(x)\,dx = \lim_{\substack{R \to +\infty \\ \varepsilon_1, \dots, \varepsilon_k \to 0^+}} \left(\int_{-R}^{c_1-\varepsilon_1} + \int_{c_1+\varepsilon_1}^{c_2-\varepsilon_2} + \dots + \int_{c_k+\varepsilon_k}^{+R} \right) f(z)\,dz}$$

(Note: the radiuses $\varepsilon_1, \varepsilon_2, \dots, \varepsilon_k$ are independent of each other!)

Finally, we shall need:

Lemma 1: If $f(z)$ has a simple pole at $z=c$ and T_ε is the arch parametrized by $z = c+\varepsilon \cdot e^{i\theta}$ with θ from θ_1 to θ_2, then

$$\lim_{\varepsilon \to 0^+} \int_{T_\varepsilon} f(z)\,dz = i \cdot (\theta_2 - \theta_1) \cdot Res_c(f(z))$$

[Major application will be to the arches γ_2, γ_2^- as above:

$$\int_{\gamma_2^-} f(z)\,dz = i \cdot (0-(-\pi)) \cdot Res_c f = i\pi \cdot Res_c(f)$$

$$\int_{\gamma_2} f(z)\,dz = i \cdot (\underset{\uparrow\, \theta_2}{0} - \underset{\uparrow\, \theta_1}{\pi}) \cdot Res_c f = -i\pi \cdot Res_c(f)$$

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Proof of Lemma

$$f(z) = \frac{a_{-1}}{z-c} + \underbrace{a_0 + a_1(z-c) + a_2(z-c)^2 + \ldots}_{g(z) \text{ - regular at } c}$$ - Laurent series expansion in small $D_\varepsilon(c) \setminus \{c\}$

$g(z)$ is ~~there~~ bounded on $D_\varepsilon(c)$, i.e $|g(z)| \le M \ \forall z \in D_\varepsilon(c)$

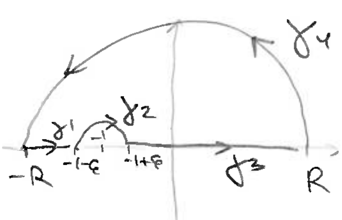
$$\Rightarrow \left| \int_{T_\varepsilon} g(z)dz \right| \le M \cdot \text{length}(T_\varepsilon) = M \cdot \varepsilon \cdot |\theta_2 - \theta_1| \xrightarrow{\varepsilon \to 0^+} 0$$

On the other hand, $\displaystyle\int_{T_\varepsilon} \frac{1}{z-c}dz = \log_\tau(z-c)\Big|_{z=c+\varepsilon e^{i\theta_1}}^{z=c+\varepsilon e^{i\theta_2}}$ with τ not b/w θ_1 & θ_2

$\quad = i(\theta_2 - \theta_1)$ - independent of ε

So: $$\int_{T_\varepsilon} f(z)dz \xrightarrow{\varepsilon \to 0^+} i(\theta_2 - \theta_1)a_{-1} = i(\theta_2 - \theta_1)\operatorname{Res}_c(f(z))$$

Ex 1: $\displaystyle \text{p.v.} \int_{-\infty}^{+\infty} \frac{e^{i\cdot 2x}}{x+1}dx = ?$



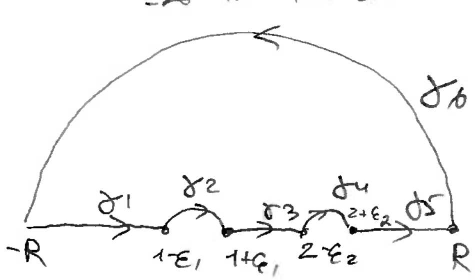
$$\left(\int_{\gamma_1} + \int_{\gamma_2} + \int_{\gamma_3} + \int_{\gamma_4} \right) \frac{e^{i\cdot 2z}}{z+1}dz = 0$$ as it's analytic inside

But: $\displaystyle\int_{\gamma_4} \frac{e^{i\cdot 2z}}{z+1}dz \xrightarrow{R \to +\infty} 0$ by Jordan's lemma

$$\int_{\gamma_2} \frac{e^{i\cdot 2z}}{z+1}dz \xrightarrow{\varepsilon \to 0^+} i(-\pi)\cdot e^{i\cdot 2(-1)} = -\pi i e^{-2i}$$

So: $\displaystyle \text{p.v.} \int_{-\infty}^{+\infty} \frac{e^{i\cdot 2x}}{x+1}dx = \pi i e^{-2i}$

Ex 2: $\displaystyle \text{p.v.} \int_{-\infty}^{+\infty} \frac{e^{ix}}{(x-1)(x-2)}dx = ?$



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$$\left(\int_{\gamma_1}+\int_{\gamma_2}+\int_{\gamma_3}+\int_{\gamma_4}+\int_{\gamma_5}+\int_{\gamma_6}\right)\frac{e^{iz}\,dz}{(z-1)(z-2)}=0 \quad \text{by Cauchy Thm.}$$

But:

$$\int_{\gamma_6}\frac{e^{iz}}{(z-1)(z-2)}dz \longrightarrow 0 \quad \text{as } R\to+\infty$$

$$\int_{\gamma_2}\frac{e^{iz}}{(z-1)(z-2)}dz \longrightarrow -i\pi\left.\frac{e^{iz}}{z-2}\right|_{z=1}=i\pi e^{i} \quad \text{as } \varepsilon_1\to0^+$$

$$\int_{\gamma_4}\frac{e^{iz}}{(z-1)(z-2)}dz \longrightarrow -i\pi\cdot\left.\frac{e^{iz}}{z-1}\right|_{z=2}=-i\pi e^{2i} \quad \text{as } \varepsilon_2\to0^+$$

So:

$$\text{p.v.}\int_{-\infty}^{+\infty}\frac{e^{ix}}{(x-1)(x-2)}dx = i\pi\left(e^{2i}-e^{i}\right)$$

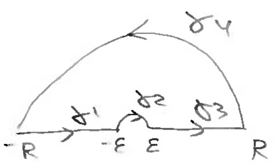
Ex 3: $I=\int_0^{+\infty}\frac{\sin 2x}{x}dx=?$

First, as in previous lectures, we note that $\frac{\sin(-2x)}{-x}=\frac{\sin 2x}{x}$, hence:

$$2I=\text{p.v.}\int_{-\infty}^{+\infty}\frac{\sin 2x}{x}dx$$

Second, note that $\sin(2x)=\operatorname{Im}\left(e^{i\cdot2x}\right)$

$$\Rightarrow \boxed{I=\frac{1}{2}\operatorname{Im}\left(\text{p.v.}\int_{-\infty}^{+\infty}\frac{e^{i2x}}{x}dx\right)}$$



$$\left(\int_{\gamma_1}+\int_{\gamma_2}+\int_{\gamma_3}+\int_{\gamma_4}\right)\frac{e^{i\cdot2z}}{z}dz=0$$

$$\int_{\gamma_4}\frac{e^{i\cdot2z}}{z}dz \longrightarrow 0 \quad \text{as } R\to+\infty \text{ by Jordan's lemma}$$

$$\int_{\gamma_2}\frac{e^{i\cdot2z}}{z}dz=-i\pi\cdot e^{i\cdot2\cdot0}=-i\pi$$

$$\Rightarrow \text{p.v.}\int_{-\infty}^{+\infty}\frac{e^{i2x}}{x}dx=i\pi \Rightarrow \operatorname{Im}(\dots)=\pi \quad\Rightarrow I=\pi/2$$

[**Remark**: Another direct way would be to integrate $\frac{e^{i\cdot2z}}{z}$ as above and $\frac{e^{i\cdot(-2z)}}{z}$ over $\gamma_1+\bar{\gamma_2}+\gamma_3+\bar{\gamma_4}$, and use $\sin 2z=\frac{e^{i\cdot2z}-e^{i(-2z)}}{2i}$