Lecture #32

- Key estimates from last week

1) $\int\limits_{|z|=R} \frac{P(z)}{Q(z)} \xrightarrow{R\to+\infty} 0$ if $\deg Q \geq \deg P + 2$.

2) $\int\limits_{\gamma_R^+ = \text{upper half circle}} \frac{P(z)}{Q(z)} e^{imz} \xrightarrow{R\to+\infty} 0$ if $m \in \mathbb{R}_{>0}$

3) $\int\limits_{\gamma_R^- = \text{lower half circle}} \frac{P(z)}{Q(z)} e^{imz} \xrightarrow{R\to+\infty} 0$ if $m \in \mathbb{R}_{<0}$

$\Big\}$ Jordan's lemma

4) $\int\limits_{S_\varepsilon^+(c)} f(z)\,dz \xrightarrow{\varepsilon\to 0^+} -\pi i \cdot \mathrm{Res}_c(f(z))$ if c-simple pole of $f(z)$

5) $\int\limits_{S_\varepsilon^-(c)} dz \xrightarrow{\varepsilon\to 0^+} \pi i \cdot \mathrm{Res}_c(f(z))$ — " —

$\Big\}$ Lecture 31

Recall: $\int\limits_{-\infty}^{+\infty} \frac{\sin x}{x}\,dx = \pi$ similarly to Ex 3 of Lecture 31

BUT $\int\limits_{-\infty}^{+\infty} \left|\frac{\sin x}{x}\right| dx$ doesn't exist!

- Today: § 6.6 = Integrals of multivalued functions

Ex 1: $I = \int\limits_0^{+\infty} \frac{1}{\sqrt{x}\,(x+1)}\,dx$

Note: 1) This function $\frac{1}{\sqrt{x}(x+1)}$ is not even, so we cannot reduce to $\frac{1}{2}\,\mathrm{p.v.}\int\limits_{-\infty}^{+\infty}$

2) Once extended to $\mathbb{C}$, we encounter a multivalued function $z^{1/2} = e^{\frac{1}{2}\log z}$.

Let me first illustrate the approach slightly different from the one in the textbook.

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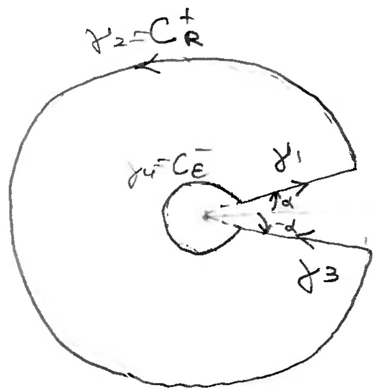
Solution of Ex 1

Pick small $\varepsilon>0$, big $R>0$, and small $\alpha>0$ and consider the contour



ray at angle $-\alpha$

$$\text{Contour } \gamma = \gamma_1 + \underbrace{C_R^+}_{\substack{\text{part}\\ \text{of circle}\\ \text{of radius } R}} + \gamma_3 + \underbrace{C_\varepsilon^-}_{\substack{\text{negatively}\\ \text{oriented part}\\ \text{of circle of}\\ \text{radius } \varepsilon}}$$

ray at angle α b/w circles

We also consider the function

$$f(z) = \frac{1}{(z+1)e^{\frac{1}{2}\log_0 z}}$$ which is analytic on γ and inside

$$\log_0 z = \ln|z| + i\cdot \underbrace{\arg_0 z}_{0<\cdots<2\pi}$$

Step 1 (Cauchy Residue Thm)

$$\int_\gamma f(z)dz = 2\pi i \cdot \operatorname{Res}_{-1} f(z) = 2\pi i \cdot \frac{1}{e^{\frac{1}{2}\log_0(-1)}} = 2\pi i \cdot \frac{1}{e^{\frac{1}{2}\cdot\pi i}} = 2\pi$$

Step 2: $$\left|\int_{C_R^+} f(z)\,dz\right| \le 2\pi R \cdot \frac{1}{R^{1/2}(R-1)} \xrightarrow{R\to+\infty} 0$$

Step 3: $$\left|\int_{C_\varepsilon^-} f(z)dz\right| \le 2\pi\varepsilon \cdot \frac{1}{\varepsilon^{1/2}(1-\varepsilon)} \xrightarrow{\varepsilon\to0^+} 0$$

Step 4: Parametrize γ_1 as $t\cdot e^{i\alpha}$ with $\varepsilon\le t\le R$ to get

$$\int_{\gamma_1} f(z)dz = \int_\varepsilon^R \frac{1}{(te^{i\alpha}+1)e^{\frac{1}{2}\log_0(te^{i\alpha})}}e^{i\alpha}dt \xrightarrow[\varepsilon, R\text{ fixed}]{\alpha\to0^+} \int_\varepsilon^R \frac{1}{(t+1)\sqrt{t}}dt$$

Step 5: Parametrize $-\gamma_3$ by $t\cdot e^{i(2\pi-\alpha)}$ with $\varepsilon\le t\le R$ to get

$$\int_{\gamma_3} f(z)dz = -\int_{-\gamma_3} f(z)dz = -\int_\varepsilon^R \frac{1}{(te^{i(2\pi-\alpha)}+1)e^{\frac{1}{2}\log_0(te^{i(2\pi-\alpha)})}}e^{i(2\pi-\alpha)}dt$$

$$\Big\downarrow \alpha\to0^+$$

$$-\int_\varepsilon^R \frac{1}{(t+1)\sqrt{t}\cdot e^{\pi i}}dt = \int_\varepsilon^R \frac{1}{(t+1)\sqrt{t}}dt$$

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(Continuation)

Thus, first taking the limit $\alpha \to 0^+$, we get

$$\int_{C_R^+} f(z)\,dz + \int_{C_\varepsilon^-} f(z)\,dz + 2\int_\varepsilon^R \frac{1}{\sqrt{t}\,(t+1)}\,dt = 2\pi.$$

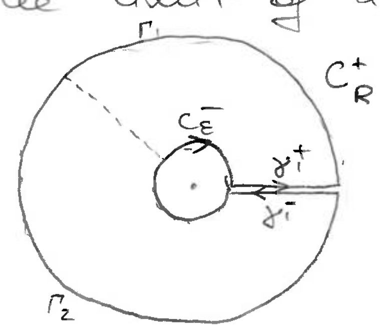
Now, taking limit $R \to +\infty$, $\varepsilon \to 0^+$, and using Steps 2-3, we get

$$2\int_0^{+\infty} \frac{1}{\sqrt{t}\,(t+1)}\,dt = 2\pi$$

$$\Downarrow$$

$$I = \pi$$

A different viewpoint to carry out the same computation is presented in §6.6 of the textbook, so let's sketch it.
Consider the limit of above contour with $\alpha = 0$:



$$f(z) = \frac{1}{z+1} \cdot e^{-\frac{1}{2}\log_0 z}$$

Key: γ_1^+ & γ_1^- denote the same line segment $[\varepsilon, R]$ in $\mathbb{R}$, but the value of $f(z)$ on them is obtained in a limit as $z \in \mathcal{H}$ = upper half plane or $z \in (-\mathcal{H})$ = lower half plane

So: $f(z) \longrightarrow \dfrac{1}{\sqrt{x}\,(x+1)}$ as $z \to x$ in $\mathcal{H}$

$f(z) \longrightarrow -\dfrac{1}{\sqrt{x}\,(x+1)}$ as $z \to x$ in $(-\mathcal{H})$

Also: While the above contour is not exactly the one to which Cauchy Residue Theorem applies, we can split along dashed arrow to get two closed positively oriented contours Γ_1 & Γ_2, noticing that on Γ_1: $\log_0 z = \log_{-\delta}(z)$ for small $\delta > 0$
on Γ_2: $\log_0 z = \log_\delta(z)$ for small $\delta > 0$

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Applying Cauchy's Residue Theorem to $\int_{\Gamma_1}$ & $\int_{\Gamma_2}$ and adding results will give us the same computation as in above proof after $\alpha \to 0^+$.

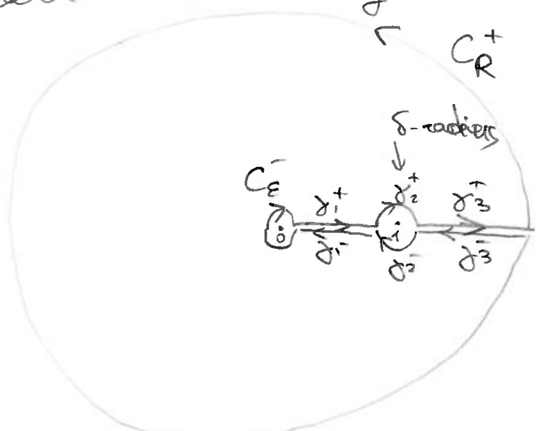
Let's illustrate this approach in our next exercise:

<u>Ex2</u>: $I = \text{p.v.} \int_0^{+\infty} \frac{1}{x^\lambda (x-1)} dx$ with $0<\lambda<1$

↑ <u>note</u>: p.v. is due to singularity at $x=1$.

By definition $I = \lim_{\substack{\varepsilon \to 0^+ \\ \delta \to 0^+ \\ R \to +\infty}} \left[\left(\int_\varepsilon^{1-\delta} + \int_{1+\delta}^{R} \right) \frac{1}{x^\lambda (x-1)} dx \right]$

We shall use the following closed contour



$\gamma = \gamma_1^+ + \gamma_2^+ + \gamma_3^+ + C_R^+ + \gamma_3^- + \gamma_2^- + \gamma_1^- + C_\varepsilon^-$

We take the following branch

$f(z) = \frac{1}{(z-1)\, e^{\lambda \log_0 z}}$

<u>Step1</u>: By the analogue of Cauchy's Residue Thm (see above discussion)

$$\int_\gamma f(z)\,dz = 0$$

<u>Step2</u>: $\left(\int_{\gamma_1^+} + \int_{\gamma_3^+} \right) f(z)\,dz = \left(\int_\varepsilon^{1-\delta} + \int_{1+\delta}^{R} \right) \frac{1}{x^\lambda (x-1)} dx$

<u>Step3</u>: $\left(\int_{\gamma_1^-} + \int_{\gamma_3^-} \right) f(z)\,dz = - e^{-2\pi i \lambda} \cdot \left(\int_{\gamma_1^+} + \int_{\gamma_3^+} \right) f(z)\,dz$

minus due to opposite orientation

$e^{-2\pi i \lambda}$ due to $\frac{1}{e^{\lambda \log_0 z}} \to \frac{1}{e^{\lambda \ln x + \lambda \cdot 2\pi i}}$ as $z \to x \in \mathbb{R}_{>0}$ in $-\mathcal{H}$.

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(Continuation)

Step 4: $\left|\int_{C_R^+} f(z)dz\right| \leq 2\pi R \cdot \frac{1}{R^\lambda (R-1)} \xrightarrow{R\to+\infty} 0$ as $\lambda>0$

Step 5: $\left|\int_{C_\varepsilon^-} f(z)dz\right| \leq 2\pi\varepsilon \cdot \frac{1}{\varepsilon^\lambda (1-\varepsilon)} \xrightarrow{\varepsilon\to 0^+} 0$ as $\lambda<1$

Step 6: On γ_2^+: $\log_0 z = \operatorname{Log} z$ — principal logarithm $\Rightarrow f(z) = \frac{1}{(z-1)e^{\lambda \operatorname{Log} z}}$
which has simple pole at 1 $\overset{\text{Lecture 31}}{\Rightarrow}$ $\int_{\gamma_2^+} f(z)dz \xrightarrow{\delta\to 0^+} -i\pi \cdot \operatorname{Res}_1(f(z))$

But $\operatorname{Res}_1\left(\frac{1}{(z-1)e^{\lambda \operatorname{Log} z}}\right) = \frac{1}{e^{\lambda \operatorname{Log}(1)}} = 1$

So: $\int_{\gamma_2^+} f(z)dz \xrightarrow{\delta\to 0^+} -i\pi$

Step 7: On γ_2^-: $f(z) = e^{-2\pi i\lambda} \cdot \frac{1}{(z-1)e^{\lambda \operatorname{Log} z}}$

$\overset{\text{analogously}}{\Longrightarrow}$ $\int_{\gamma_2^-} f(z)dz \xrightarrow{\delta\to 0^+} -i\pi \cdot e^{-2\pi i\lambda}$

Combining all these steps and taking the limit $\varepsilon\to 0^+, \delta\to 0^+, R\to+\infty$:

$$(1-e^{-2\pi i\lambda})I = i\pi + i\pi e^{-2\pi i\lambda}$$

$$\Downarrow$$

$$I = i\pi \cdot \frac{1+e^{-2\pi i\lambda}}{1-e^{-2\pi i\lambda}} = \pi \cdot \frac{\cos(\pi\lambda)}{\sin(\pi\lambda)} = \underbrace{\pi\cot(\pi\lambda)}_{\text{clearly real as it should be!}}$$

Next time:

Ex 3: $I = \int_0^{+\infty} \frac{dx}{(x+1)(x^2+2x+2)}$