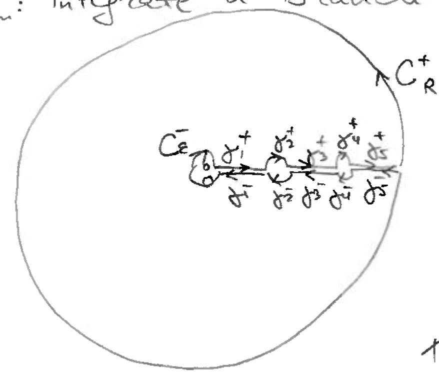
Lecture #33

Last time: $\text{p.v.}\int_0^{+\infty} x^\lambda \frac{P(x)}{Q(x)}\,dx$ with λ – not an integer

Key: integrate a branch $f(z) = e^{\lambda \log_0 z} \cdot \frac{P(z)}{Q(z)}$ over the contour γ that looks as

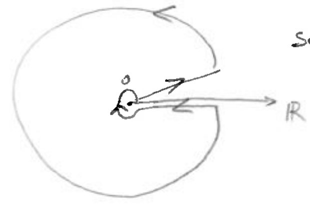


moving clockwise around circle of radius ε at the origin, then moving from $+\varepsilon$ to $+R$ "on top" of line segments ($\gamma_1^+, \gamma_3^+, \gamma_5^+$) while moving on top half circles (γ_2^+, γ_4^+) around positive real poles of $\frac{P(z)}{Q(z)}$, then moving counterclockwise around big circle of radius R and then moving back from $+R$ to $+\varepsilon$ but "on bottom" of the segments ($\gamma_5^-, \gamma_3^-, \gamma_1^-$) and moving on bottom half circles (γ_4^-, γ_2^-) around real positive singularities of $\frac{P(z)}{Q(z)}$.

* Other multivalued functions we encountered in class are $\arg z$ & $\log z$. For example as we choose the branch $\log_0 z$, which is discontinuous on $\mathbb{R}_{\geq 0}$ we get

$$\boxed{\log_0 z \longrightarrow \begin{cases} \ln x, & \text{if } z \to x \in \mathbb{R}_{>0} \ (\text{with } z \in \mathcal{H}) \\ \ln x + 2\pi i, & \text{if } z \to x \in \mathbb{R}_{>0} \ (\text{with } z \in -\mathcal{H}) \end{cases}} \qquad (*)$$

Remark: Another approach suggested last time was to integrate over



so that above $f(z)$ is actually analytic on & inside the contour, but this doesn't work well when $\frac{P(z)}{Q(z)}$ has real positive poles (as above contour doesn't account for them)

Ex1: Evaluate $I = \int_0^{+\infty} \frac{dx}{(x+1)(x^2+2x+2)}$ ← Example 4 in Section 6.6 of your book.

Warning: a) Usually when integrating rational f-s one tries to first use partial fraction decomposition. However, it's not going to work here as e.g. $\int_0^{+\infty} \frac{1}{x+1}\,dx$ is not well-defined while the above I is (since $\frac{1}{(x+1)(x^2+2x+2)} \sim \frac{1}{x^3}$ for $x \gg 0$ and $\int_1^{+\infty} \frac{dx}{x^3}$ exists).

b) Also we cannot reduce to $\frac{1}{2}\,\text{p.v.}\int_{-\infty}^{+\infty} \text{-}\text{''}\text{-}\,dx$ as f-n is not even!

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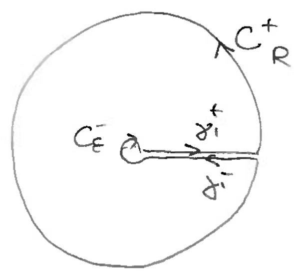
Solution of Ex1

▶ Integrate a different function

$$f(z)=\frac{\log_0 z}{(z+1)(z^2+2z+2)}$$

over the contour



$$\gamma=\gamma_1^+ + C_R^+ + \gamma_1^- + C_\varepsilon^-$$

Key:

$$\int_{\gamma_1^+} f(z)dz=\int_\varepsilon^R \frac{\ln x}{(x+1)(x^2+2x+2)}dx$$

$$\int_{\gamma_1^-} f(z)dz=-\int_\varepsilon^R \frac{\ln x+2\pi i}{(x+1)(x^2+2x+2)}dx$$

due to (*) on page 1 $\Rightarrow$

$$\Rightarrow \left(\int_{\gamma_1^+}+\int_{\gamma_1^-}\right)f(z)dz=-2\pi i\cdot\int_\varepsilon^R \frac{1}{(x+1)(x^2+2x+2)}dx$$

as $\varepsilon\to 0^+$, $R\to+\infty$ this gives us the desired I.

As always: expect $\lim\limits_{R\to+\infty}\int_{C_R^+} f(z)dz=0$, $\lim\limits_{\varepsilon\to 0^+}\int_{C_\varepsilon^-} f(z)dz=0$. Let's prove these:

$$\left|\int_{C_R^+} f(z)dz\right|\le 2\pi R\cdot\frac{\sqrt{(\ln R)^2+(2\pi)^2}}{(R-1)(R^2-2R-2)}\xrightarrow{R\to+\infty}0 \quad \text{as} \quad \frac{\ln R}{R^2}\xrightarrow{R\to+\infty}0$$

$$\left|\int_{C_\varepsilon^-} f(z)dz\right|\le 2\pi\varepsilon\cdot\frac{\sqrt{(\ln\varepsilon)^2+(2\pi)^2}}{(1-\varepsilon)(2-2\varepsilon-\varepsilon^2)}\xrightarrow{\varepsilon\to 0^+}0 \quad \text{as} \quad \varepsilon\cdot\ln\varepsilon\xrightarrow{\varepsilon\to0^+}0 \quad \left(\text{e.g. write } \varepsilon=e^y \text{ with } y\to-\infty\right)$$

Thus:

$$\boxed{\lim_{\substack{\varepsilon\to0^+\\ R\to+\infty}}\int_\gamma f(z)dz=-2\pi i\cdot I}$$

On the other hand, $\int_\gamma f(z)dz$ can be computed via the Cauchy Residue Theorem: note that singular points inside are -1, $-1+i$, $-1-i$, as denominator $=(z+1)(z-(-1+i))(z-(-1-i))$. Let's evaluate the corresponding residues:

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(Continuation)

$$\left.\begin{aligned}
\operatorname{Res}_{-1} f(z) &= \left.\frac{\log_0 z}{z^2+2z+2}\right|_{z=-1} = \frac{i\cdot\pi}{1} = \pi i \\
\operatorname{Res}_{-1+i} f(z) &= \left.\frac{\log_0 z}{(z+1)(z-(-1-i))}\right|_{z=-1+i} = \frac{\ln\sqrt{2}+i\cdot\frac{3\pi}{4}}{i\cdot 2i} = -\frac{\ln\sqrt{2}}{2} - \frac{3\pi}{8}i \\
\operatorname{Res}_{-1-i} f(z) &= \left.\frac{\log_0 z}{(z+1)(z-(-1+i))}\right|_{z=-1-i} = \frac{\ln\sqrt{2}+i\cdot\frac{5\pi}{4}}{(-i)\cdot(-2i)} = -\frac{\ln\sqrt{2}}{2} - \frac{5\pi}{8}i
\end{aligned}\right\} \Rightarrow$$

$$\Rightarrow \int_\gamma f(z)\,dz = 2\pi i\left(\pi i - \frac{\ln\sqrt{2}}{2} - \frac{3\pi}{8}i - \frac{\ln\sqrt{2}}{2} - \frac{5\pi}{8}i\right) = 2\pi i\cdot(-\ln\sqrt{2})$$

So: $-2\pi i\cdot I = -2\pi i\ln\sqrt{2} \Rightarrow I = \ln\sqrt{2}$

* * *

Before moving to the last section §6.7 we shall discuss an important technique for summation of series following p. 327 of your book (see Exercise 14)

! This material can appear on Midterm 2 or Final!

The following is the basic example of the general theory:

Ex 2: Evaluate $\sum_{k=-\infty}^{+\infty} \frac{1}{k^2+1}$ (looks like calculus problem, but usually in calculus you only get upper bounds on such sums rather than explicit values)

We now state the general result:

Theorem: Let $f(z) = \frac{P(z)}{Q(z)}$ be a rational function with $\deg Q \geq \deg P + 2$ and without any poles at $\mathbb{Z} = \{\text{integers}\}$. Then:

$$\underbrace{\lim_{N\to+\infty} \sum_{k=-N}^{+N} f(k)}_{=\sum_{k\in\mathbb{Z}} f(k) \text{ if all of same sign}} = -\left\{\text{sum of residues of } \pi\cdot f(z)\cdot\cot(\pi z) \text{ at poles of } f(z)\right\}$$

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Solution of Ex2

Consider $f(z)=\frac{1}{z^2+1}$ which satisfies conditions of the theorem. Then:

$$\sum_{k=-\infty}^{+\infty}\frac{1}{k^2+1}=-\mathrm{Res}_i\left(\frac{\pi\cot(\pi z)}{(z-i)(z+i)}\right)-\mathrm{Res}_{-i}\left(\frac{\pi\cot(\pi z)}{(z-i)(z+i)}\right)$$

<u>Here</u>: $$\mathrm{Res}_i\left(\frac{\pi\cot(\pi z)}{(z-i)(z+i)}\right)=\frac{\pi\cot(\pi z)}{z+i}\Big|_{z=i}=\frac{\pi}{2i}\cdot\frac{\cos(\pi i)}{\sin(\pi i)}=\frac{\pi}{2i}\cdot\frac{\frac{e^{\pi i^2}+e^{-\pi i^2}}{2}}{\frac{e^{\pi i^2}-e^{-\pi i^2}}{2i}}=\frac{\pi}{2}\cdot\frac{e^{-\pi}+e^{\pi}}{e^{-\pi}-e^{\pi}}$$

$$\mathrm{Res}_{-i}\left(\frac{\pi\cot(\pi z)}{(z-i)(z+i)}\right)=\frac{\pi\cot(\pi z)}{z-i}\Big|_{z=-i}=\frac{\pi}{2i}\cdot\frac{\cos(-\pi i)}{\sin(-\pi i)}$$

<u>Thus</u>: $$\sum_{k=-\infty}^{+\infty}\frac{1}{k^2+1}=-2\cdot\frac{\pi}{2}\cdot\frac{e^{-\pi}+e^{\pi}}{e^{-\pi}-e^{\pi}}=\pi\cdot\frac{e^{\pi}+e^{-\pi}}{e^{\pi}-e^{-\pi}}=\pi\cdot\coth(\pi)$$

Let's now prove the above theorem.

Proof of Theorem

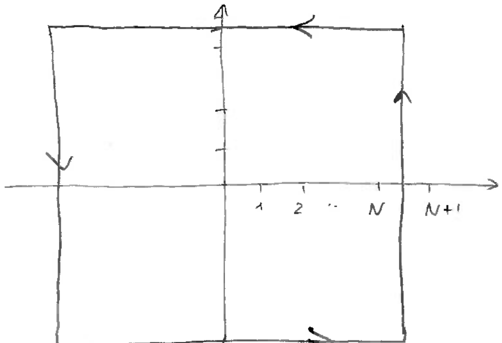
Consider the function $g(z)=\pi\cdot\cot(\pi z)\cdot f(z)$. Its singularities are:

- poles of $f(z)$
- simple poles of $\cot(\pi z)$ which are at $\mathbb{Z}=\{\text{integers}\}$

Moreover, for any integer $k\in\mathbb{Z}$:

$$\boxed{\mathrm{Res}_k\, g(z)=\mathrm{Res}_k\left(\frac{\pi\cdot\cos(\pi z)\cdot f(z)}{\sin(\pi z)}\right)=\frac{\pi\cdot\cos(\pi z)\cdot f(z)}{(\sin(\pi z))'}\Big|_{z=k}=f(k)}$$

Let's now pick a contour Γ_N which is a positively oriented square with vertices $(N+\frac{1}{2})(1+i)$, $(N+\frac{1}{2})(-1+i)$, $(N+\frac{1}{2})(-1-i)$, $(N+\frac{1}{2})(1-i)$



$$\boxed{\text{Claim: There is a constant } M\in\mathbb{R}_{>0} \text{ such that } |\pi\cot\pi z|\le M\ \ \forall z\in\Gamma_N\ \ \forall \text{ integer } N.}$$

$\Downarrow$

$\lim_{N\to\infty}\int_{\Gamma_N} g(z)dz=0$ as $\left|\int_{\Gamma_N} g(z)dz\right|\le M\cdot 4\cdot(2N+1)\cdot\frac{K}{N^2}\xrightarrow[N\to+\infty]{}0$

with K - some other constant (<u>key</u>: $\deg Q\ge\deg P+2$)

Applying Cauchy's Residue theorem now implies the result.

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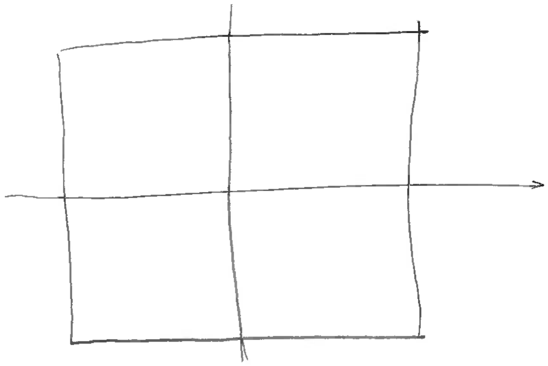
(Continuation of Proof)

It thus remains to prove the Claim from p. 4



Case 1: $z = x+yi$ with $y > \frac{1}{2}$

Then $|\cot(\pi z)| = \left| \frac{e^{i\pi z} + e^{-i\pi z}}{e^{i\pi z} - e^{-i\pi z}} \right| \overset{\text{triangular inequalities}}{\leq} \frac{|e^{i\pi z}| + |e^{-i\pi z}|}{|e^{-i\pi z}| - |e^{i\pi z}|} = \frac{1+e^{-2\pi y}}{1-e^{-2\pi y}}$

But $h(y) := \frac{1+e^{-2\pi y}}{1-e^{-2\pi y}}$ is decreasing on $[\frac{1}{2}, +\infty)$ as

$$h'(y) = \frac{e^{-2\pi y}\cdot(-2\pi)(1-e^{-2\pi y}) - (1+e^{-2\pi y})\cdot e^{-2\pi y}\cdot(-(-2\pi))}{(1-e^{-2\pi y})^2} = \frac{-4\pi e^{-2\pi y}}{(1-e^{-2\pi y})^2} < 0$$

So: $|\cot(\pi z)| \leq h(\frac{1}{2}) = \frac{1+e^{-\pi}}{1-e^{-\pi}}$

Case 2: $z = x+yi$ with $y < -\frac{1}{2}$

Same bound (as $|\cot(\pi z)| = |\cot(-\pi z)|$): $|\cot \pi z| \leq \frac{1+e^{-\pi}}{1-e^{-\pi}}$

Case 3: $z = N + \frac{1}{2} + yi$ with $-\frac{1}{2} \leq y \leq \frac{1}{2}$

Then $|\cot(\pi z)| = |\cot(\frac{\pi}{2} + i\cdot\pi y)| = |\tan(i\cdot\pi y)| = |\tanh(\pi y)|$ with $\tanh\theta = \frac{e^{2\theta}-1}{e^{2\theta}+1}$

But $\frac{d}{d\theta}\tanh\theta = \frac{e^{2\theta}\cdot 2(e^{2\theta}+1) - e^{2\theta}\cdot 2(e^{2\theta}-1)}{(e^{2\theta}+1)^2} = \frac{4e^{2\theta}}{(e^{2\theta}+1)^2} > 0 \Rightarrow \tanh$ - increases

So: $|\cot(\pi z)| \leq |\tanh\frac{\pi}{2}| = \frac{e^{\pi}-1}{e^{\pi}+1}$

Case 4: $z = -N - \frac{1}{2} + yi$ with $-\frac{1}{2} \leq y \leq \frac{1}{2}$

Same bound (as $|\cot(\pi z)| = |\cot(-\pi z)|$): $|\cot \pi z| \leq \frac{e^{\pi}-1}{e^{\pi}+1}$

! For interested students: modify thm to prove famous $\sum_{k=1}^{+\infty} \frac{1}{k^2} = \pi^2/6$.