Lecture #34 §6.7: Argument Principle and Rouché's theorem

Def.: Function $f(z)$ is *meromorphic* at domain D if at every point $z \in D$ it's either analytic or has a pole

Today: $\begin{cases} D = \text{inside of some simple closed curve } \gamma \\ f\text{- analytic and nonzero on } \gamma \text{, and meromorphic on } D. \end{cases}$

Claim: In this setup, f has a finite number of zeroes & poles in D

(discuss in class! key: Bolzano - Weierstrass theorem)

Let's introduce

$$\begin{cases} N_0(f) = \text{sum of orders of all zeroes of } f(z) \text{ inside } D \\ N_p(f) = \text{sum of orders of all poles of } f(z) \text{ inside } D \end{cases}$$

Our first main result today is going to express $\oint_\gamma \frac{f'(z)}{f(z)} dz$ via $N_0(f)$, $N_p(f)$.

First, let's look at the "baby example".

Ex 1: Evaluate $\oint_{|z|=3} \frac{f'(z)}{f(z)} dz$ for $f(z) = \frac{(z-1)^2 (z-2)^3 (z-5)^{10}}{(z^2+1)^3 (z^2+4)^5}$

Calculus $\Rightarrow$ $\frac{f'(z)}{f(z)} = \frac{2}{z-1} + \frac{3}{z-2} + \frac{10}{z-5} - \frac{3}{z-i} - \frac{3}{z+i} - \frac{5}{z-2i} - \frac{5}{z+2i}$

As $\oint_{|z|=3} \frac{dz}{z-c} = \begin{cases} 2\pi i & \text{if } |c|<3 \\ 0 & \text{if } |c|>3 \end{cases}$ $\Rightarrow$ $\oint_{|z|=3} \frac{f'(z)}{f(z)} dz = 2\pi i \left(\underbrace{(2+3)}_{N_0(f)} - \underbrace{(3+3+5+5)}_{N_p(f)}\right) = -22\pi i$

Now we are ready to state the first theorem for today:

Theorem 1 ("Argument Principle"): If f is analytic and nonzero at each point of a simple closed positively oriented contour Γ and meromorphic inside Γ, then

$$\oint_\Gamma \frac{f'(z)}{f(z)} dz = 2\pi i \left(N_0(f) - N_p(f) \right)$$

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Proof of Theorem 1

- Singular points of $\frac{f'(z)}{f(z)}$ are either zeroes or poles of $f(z)$

Case 1: If z_0 - zero of $f(z)$ inside Γ, then looking at Taylor expansion of $f(z)$ around z_0, we get $f(z) = (z-z_0)^n \underbrace{(\overset{\neq 0}{a_n} + a_{n+1}(z-z_0) + \dots)}_{\tilde{f}(z)}$

where n = order of zero at z_0

Note: $\tilde{f}(z)$ analytic in nbhd of z_0 and $\tilde{f}(z_0) = a_n \neq 0$

Hence: $$\frac{f'(z)}{f(z)} = \frac{n(z-z_0)^{n-1}\cdot\tilde{f}(z) + (z-z_0)^n\tilde{f}'(z)}{(z-z_0)^n\tilde{f}(z)} = \frac{n}{z-z_0} + \frac{\tilde{f}'(z)}{\tilde{f}(z)}$$

As $\frac{\tilde{f}'(z)}{\tilde{f}(z)}$ is analytic in $D_\varepsilon(z_0)$ for some $\varepsilon > 0$, we get

$$\oint_{C_\varepsilon(z_0)} \frac{f'(z)}{f(z)}dz = n\cdot 2\pi i$$

Case 2: If z_0 - pole of $f(z)$ inside Γ, then looking at the Laurent series expansion in small punctured disk centered at z_0, we get

$f(z) = (z-z_0)^{-n}\cdot \underbrace{(\overset{\neq 0}{a_{-n}} + a_{1-n}(z-z_0) + \dots)}_{\tilde{f}(z) \text{ - analytic at nbhd of } z_0}$ with n = order of pole at z_0

Arguing as above, we then get $\oint_{C_\varepsilon(z_0)} \frac{f'(z)}{f(z)}dz = 2\pi i\cdot(-n)$

Now: As $\frac{f'(z)}{f(z)}$ is analytic inside $\Gamma \setminus \{\text{zeros \& poles}\}$, we get

$$\oint_\Gamma \frac{f'(z)}{f(z)}dz = \sum_{z_0 \text{ - zero or pole}} \oint_{C_\varepsilon(z_0)} \frac{f'(z)}{f(z)}dz = 2\pi i\,(N_0(f) - N_p(f))$$

∎

Let's now get a more geometric interpretation of above integral

Recall (from calculus): If $f: [a,b] \to \mathbb{R}_{>0}$ then $\boxed{\frac{f'(x)}{f(x)} = \frac{d}{dx}\ln(f(x))}$

"logarithmic derivative"

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Now, in the complex setup, we need to choose a branch of log which may not be possible for all $\{f(z) \mid z \in \gamma\}$. However, we can clearly break $\gamma = \gamma_1 + \gamma_2 + \dots + \gamma_N$ into several subarcs $\{\gamma_j\}_{j=1}^N$ (say between points y_j & y_{j+1} on γ) so that the argument of $f(z)$ varies by less than 2π for $z \in \gamma_j$ which then allows to pick a branch of log (separately for each γ_j). By Fundamental theorem of calculus:

$$\int_{\gamma_j} \frac{f'(z)}{f(z)}\,dz = \left\{ \ln|f(z)| + i \arg_{\tau_j} f(z) \right\} \Big|_{y_j}^{y_{j+1}}$$ with τ_j determining the branch

$$\Downarrow$$

$$\oint_\gamma \frac{f'(z)}{f(z)}\,dz = \big(\ln|f(y_2)| - \ln|f(y_1)|\big) + \big(\ln|f(y_3)| - \ln|f(y_2)|\big) + \dots + \big(\ln|f(y_1)| - \ln|f(y_N)|\big)$$

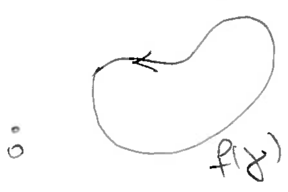
$$+ i \cdot \Delta_\gamma \arg f(z) = i \cdot \underbrace{\Delta_\gamma \arg f(z)}_{\text{"net excursion" of } \arg f(z)}$$

(key: similarly defined $\Delta_\gamma \ln|f(z)|$ is clearly zero as seen above!)

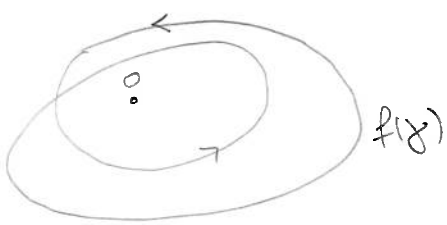
Combining with Theorem 1, we get:

$$\boxed{\text{Corollary: } \Delta_\gamma \arg f(z) = 2\pi\big(N_0(f) - N_p(f)\big)}$$

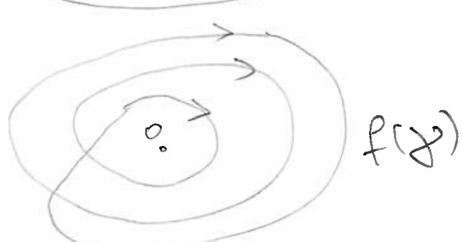
Examples:



$\Delta_\gamma \arg f(z) = 0$



$\Delta_\gamma \arg f(z) = 4\pi \Rightarrow N_0(f) \geq 2$



$\Delta_\gamma \arg f(z) = -6\pi \Rightarrow N_p(f) \geq 3$

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Theorem 2 (Rouché's Theorem): If γ - simple closed curve and f, h are analytic on γ and inside it, and $|h(z)| < |f(z)|$ for any $z \in \gamma$, then:

$$N_0(f) = N_0(f+h)$$

i.e. f and $f+h$ have the same number of zeroes inside γ

"Pictorial proof" (see pages 360-361 of the textbook): think of $f(z)$ as being the path of you and $f(z)+h(z)$ - path of your dog (with $z \in \gamma$) as you walk around the lamppost located at the origin. The condition $|h(z)| < |f(z)|$ $\forall z \in \gamma$ guarantees that the leash never extends past the lamppost. Hence:

$$\underbrace{\Delta_\gamma \arg(f(z))}_{2\pi \cdot N_0(f)} = \underbrace{\Delta_\gamma \arg(f(z)+h(z))}_{2\pi \cdot N_0(f+h)} \quad \text{(by Corollary)}$$

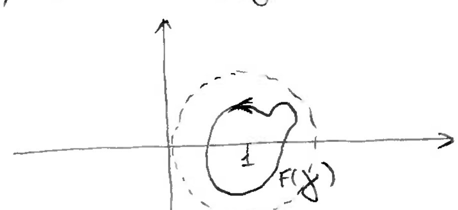
$$\Downarrow$$

$$N_0(f) = N_0(f+h)$$

Rigorous Proof of Theorem 2

Let $g = f+h$. Then $|h| < |f|$ on $\gamma \Rightarrow \left|1 - \frac{g}{f}\right| < 1$ on γ.

Consider $F(z) = \frac{g(z)}{f(z)}$, so that $F(\gamma)$ is inside $D_1(1)$:



Then the principal logarithm is well-defined on $F(\gamma)$

So: $\frac{d}{dz} \operatorname{Log} F(z) = \frac{F'(z)}{F(z)} = \frac{g'(z) \cdot f(z) - g(z) f'(z)}{f(z)^2} \cdot \frac{f(z)}{g(z)} = \frac{g'(z)}{g(z)} - \frac{f'(z)}{f(z)}$

Thus: $0 = \oint_\gamma \frac{d}{dz} \operatorname{Log} F(z)\, dz = \oint_\gamma \frac{F'(z)}{F(z)} dz = \oint_\gamma \frac{g'(z)}{g(z)} dz - \oint_\gamma \frac{f'(z)}{f(z)} dz$

$\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad$ || Argument Principle

$$2\pi i\,(N_0(g) - N_0(f))$$

This proves $N_0(f) = N_0(g)$ $\quad$ ($g = f+h$)

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The key application of Rouché's theorem is to the location of zeroes of analytic functions $g(z)$ by comparing to zeroes of simpler function $f(z)$

Example: Show that all roots of $g(z) = z^4 + 7z^2 + 3z + 5$ lie in disk $|z| < 3$

▸ We want to pick $f(z)$ to be the "leading term" for z on circle of radius 3.

Clear: $|z^4| = 81$, $|z^2| = 9$, $|z| = 3$ on this circle.

So: let's pick $f(z) = z^4$ and $h(z) = g(z) - f(z) = 7z^2 + 3z + 5$.

Then: $|h(z)| \le 7 \cdot 9 + 3 \cdot 3 + 5 = 77 < 81 = |f(z)|$ for all z on this circle

$\Downarrow$

$N_0(g) = N_0(f)$

But clearly z^4 has only one zero at $z_0 = 0$ of order 4 $\Big\} \Rightarrow N_0(g) = 4$

Since any polynomial of degree 4 has 4 roots $\Rightarrow$ all inside $D_3(0)$ ∎

Ex2: How many zeroes does $g(z) = z^3 + 8z + 25$ have on $D_2(0) = \{z \mid |z| \le 2\}$?

▸ For $|z| = 2$ have $|z^3| = 8$, $|z| = 2 \Rightarrow |z^3 + 8z| \le 24$

Thus: we should take $f(z) = 25$ and $h(z) = g(z) - f(z) = z^3 + 8z$.

Then, Rouché's thm $\Rightarrow N_0(g) = N_0(f)$. But f is a nonzero constant, hence, has no zeroes!

So: $g(z)$ has no zeroes on $D_2(0)$! ∎

Ex3: Show that all zeroes of $g(z) = z^5 - 6z^2 + 7.5$ fall in the annulus $\{1 < |z| < 2\}$.

▸ This is a combination of above Example + Ex2.

• For $\gamma = \{|z| = 2\}$, take $f(z) = z^5$, $h(z) = -6z^2 + 7.5$. Then $|h(z)| \le 31.5 < |f(z)|$ for $|z| = 2 \Rightarrow N_0(g) = N_0(f) = 5 \xRightarrow{g\text{-polynomial of deg } 5}$ all zeroes of $g(z)$ are in $D_2(0)$

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(Continuation)

- For $\gamma = \{|z|=1\}$, we take $f(z) = 7.5$ and $h(z) = z^5 - 6z^2$.

Then $|h(z)| \leq 7 < |f(z)|$ for all $|z|=1$ $\Rightarrow$ $N_0(g) = N_0(f) = 0$ $\Rightarrow$ no zeroes in $D_1(0)$.

But above inequalities also imply no zeroes on $C_1(0)$.

Combining these two steps, we get the result! ∎

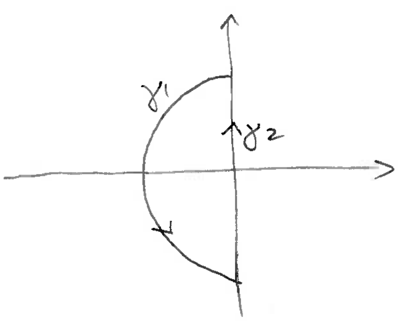
Ex 4: Prove that $z+4+3e^{2z} =: g(z)$ has precisely one zero in $D = \{z \in \mathbb{C} : \mathrm{Re}\, z \leq 0\}$

Warning: First, we note that $g(z)$ is not a polynomial (as in previous Ex)

More importantly D is not an interior of some closed curve

Note that $|e^{2z}| \leq 1$ for $z \in D$

For any $\rho > 7$, consider the half circle in D of radius ρ centered at the origin, followed up by vertical segment from $-\rho i$ to ρi.



$\gamma = \gamma_1 + \gamma_2$

Then: $|z+4| \geq 4$ on γ_2

$|z+4| \geq \rho - 4 > 3$ on γ_1

Take $f(z) = z+4$, $h(z) = 3e^{2z}$, so that $|h(z)| < |f(z)|$ for all $z \in \gamma$.

By Rouché's theorem: $N_0(g) = N_0(f) = 1$ ← single simple zero at $z_0 = -4$

Thus, $g(z)$ has a single zero inside above γ for any $\rho > 7$. Taking limit $\rho \to +\infty$, we clearly get the claim ∎

↑ indeed, if there were at least two zeroes $z_1, z_2 \in D$, then for $\rho > 7, |z_1|, |z_2|$ the above argument yields a contradiction!