

* Last time: Argument principle & Rouché's theorem

→ go over the proof of Rouché's Thm (see p.4 of Notes for Lecture 34)

Note: Applying the same proof in the case when h -analytic on & inside γ , while f -analytic on γ and meromorphic inside γ , with still $|h(z)| < |f(z)|$ for all $z \in \gamma$, we get

$$N_0(f) - N_p(f) = N_0(f+h) - N_p(f+h)$$

But if z_0 -pole of f of order m , then z_0 -also order m pole of $f+h$
'inside γ '

$\Rightarrow N_p(f) = N_p(f+h)$. So:

Claim: In the above setup, we again have $N_0(f) = N_0(f+h)$

↑ see Exercises 14-15 to §6.7 (p.365).

Hint: Rouché's theorem also provides another proof of the Fund. Thm. Algebra

Given any polynomial $p(z) = a_0 + a_1 z + \dots + a_{n-1} z^{n-1} + a_n z^n$ with $a_n \neq 0$

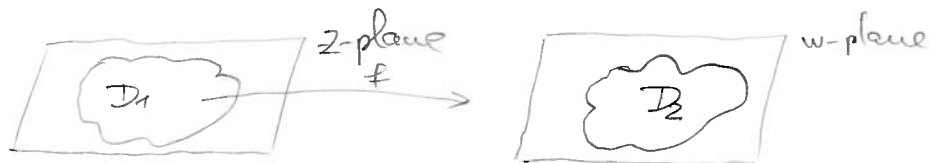
let's pick $f(z) = a_n z^n$ and $h(z) = a_0 + a_1 z + \dots + a_{n-1} z^{n-1}$. Then, as

$\lim_{z \rightarrow \infty} \left(\frac{a_{n-1}}{z} + \dots + \frac{a_1}{z^{n-1}} + \frac{a_0}{z^n} \right) = 0$, we see $\exists R > 0$ s.t. $|h(z)| < |f(z)|$

for all $z \in C_R(0) = \{z : |z| = R\}$. Then: $N_0(p) = N_0(f) = n$

Finally, §6.7 ended with the "Open Mapping" property of non-constant analytic functions, but we'll get to it at the end of today's class.

* Today: Geometric Properties of $z \mapsto w = f(z)$; §7.1-7.2.



Identifying $z \in \mathbb{C}$ with $(x, y) \in \mathbb{R}^2$, can think of f as $(u(x, y), v(x, y))$

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Theorem 1: If $f: D_1 \rightarrow D_2$ is analytic 1-to-1 and $f'(z) \neq 0 \forall z \in D_1$, then the inverse $f^{-1}: D_2 \rightarrow D_1$ is analytic and $\frac{df^{-1}}{dw}(w) = \frac{1}{\frac{df}{dz}(z)}$ with $w=f(z)$

► This was proved in [Lecture 11, Theorem 3] and crucially based on the inverse function theorem and $\det \begin{pmatrix} u_x & u_y \\ v_x & v_y \end{pmatrix} = |f'(z)|^2 \neq 0$

Theorem 2: In the above setup, if $\phi = \phi(u, v)$ is harmonic on D_2 , then $\phi \circ f$ - harmonic on D_1

► ϕ is harmonic at $w_0 = f(z_0) \Rightarrow$ on some small open disk $D_r(w_0) \subset D_2$ there is an analytic function $g: D_r(w_0) \rightarrow \mathbb{C}$ s.t. $\phi = \operatorname{Re}(g)$. But then $\phi \circ f = \operatorname{Re}(g \circ f)$ which holds true on some small disk $D_\delta(z_0) \subset D_1$ (which exists as f -continuous $\Rightarrow \exists \delta > 0$ with $D_\delta(z_0) \subseteq f^{-1}(D_r(w_0))$)

But f, g -analytic $\Rightarrow g \circ f$ -analytic $\Rightarrow \operatorname{Re}(g \circ f)$ -harmonic

Application: Solving Dirichlet problem $\nabla^2 \phi = 0$ in domain D (with specified boundary value) by constructing analytic 1-to-1 map $D \xrightarrow{f} \tilde{D}$, picking harmonic $\tilde{\phi}$ on $\tilde{D}$, and then constructing $\phi = \tilde{\phi} \circ f$.

Recall: Know explicit solutions for:

- washers, wedges, walls
- disks (Poisson's integral f.b.) & $\mathbb{H}$ = upper half plane

* "Local" properties

Theorem 3: If f -analytic at z_0 and $f'(z_0) \neq 0$, then $\exists \varepsilon > 0$ s.t. f is 1-to-1 on $D_\varepsilon(z_0)$

► One way: same as for Theorem 1 - just use inverse function theorem.

Other proof - see book, p. 378: Pick $\varepsilon > 0$ s.t. $|f'(z) - f'(z_0)| < \frac{1}{2}|f'(z_0)|$ for any $z \in D_\varepsilon(z_0)$ (here use f' -analytic $\Rightarrow$ continuous). For any two $z_1, z_2 \in D_\varepsilon(z_0)$, let γ = line segment from z_1 to z_2 , so that:

$$|f(z_1) - f(z_2)| = \left| \int_\gamma f'(z) dz \right| = \left| \underbrace{\int_\gamma f'(z_0) dz}_{= f'(z_0) \cdot (z_2 - z_1)} + \underbrace{\int_\gamma (f'(z) - f'(z_0)) dz}_{|\dots| \leq \frac{1}{2}|f'(z_0)| \cdot |z_2 - z_1|} \right| \geq \underbrace{\frac{1}{2}|f'(z_0)| \cdot |z_2 - z_1|}_0$$

Non-example: For any $n \geq 2$ -integer, the map $z \mapsto z^n$ is actually n -to-1 in any $D_\varepsilon(0) \setminus \{0\}$

Def: A map $f: D_1 \rightarrow D_2$ is called conformal at point $z_0 \in D_1$ if it preserves angles b/w curves intersecting at z_0 .



Theorem 4: If f -analytic and $f'(z_0) \neq 0$, then f -conformal at z_0

Non-example: For any $n \geq 2$ -integer, the map $z \mapsto z^n$ at $z_0 = 0$ multiplies angles by n ! (so is not conformal).

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Proof of Thm 4

► Pick any curve γ passing through z_0 and parametrize it by $z(t)$, so that $z_0 = z(t_0)$

Then, the tangent vector (a.k.a. velocity vector) is $\vec{T} = z'(t_0)$

Now, the curve $f(\gamma)$ is naturally parametrized via $f(z(t))$, and so

the corresponding tangent vector $\vec{T}$ at $w_0 = f(z_0)$ is

$$\frac{d}{dt}(f(z(t))) = f'(z(t_0)) \cdot z'(t_0) = f'(z_0) \cdot z'(t_0)$$

As $f'(z_0) \neq 0$, let's write in polar coordinates $f'(z_0) = re^{i\varphi}$. Then, we see

that $\vec{T}$ is obtained from $\vec{T}$ by rotating by angle φ and scaling by r

Thus: the angle between any two curves passing through z_0 is preserved. $\square$

*"Global" properties

Theorem 5 ("Open Mapping property"): Any non-constant analytic function maps open sets to open sets

► Pick any z_0 and first assume that $f(z_0) = 0$. As f -nonconstant, $\exists \varepsilon > 0$ such that f has no zeroes in $|z - z_0| \leq \varepsilon$ (as zeroes - isolated set).

Let $c := \min_{|z - z_0| \leq \varepsilon} |f(z)| > 0$. For any $\delta \in \mathbb{C}$ with $|\delta| < c$, apply Rouché's

Theorem to $g = C_\varepsilon(z_0)$, $f(z)$, $h(z) = -\delta$ to deduce $N_0(f - \delta) = N_0(f) > 0$

Hence: f takes value δ inside $C_\varepsilon(z_0) \Rightarrow f(D_\varepsilon(z_0))$ contains $D_c(0)$.

Now, if $f(z_0) = w_0 \neq 0$, then just apply above argument to

$$F(z) = f(z) - w_0.$$

The last theorem for today is:

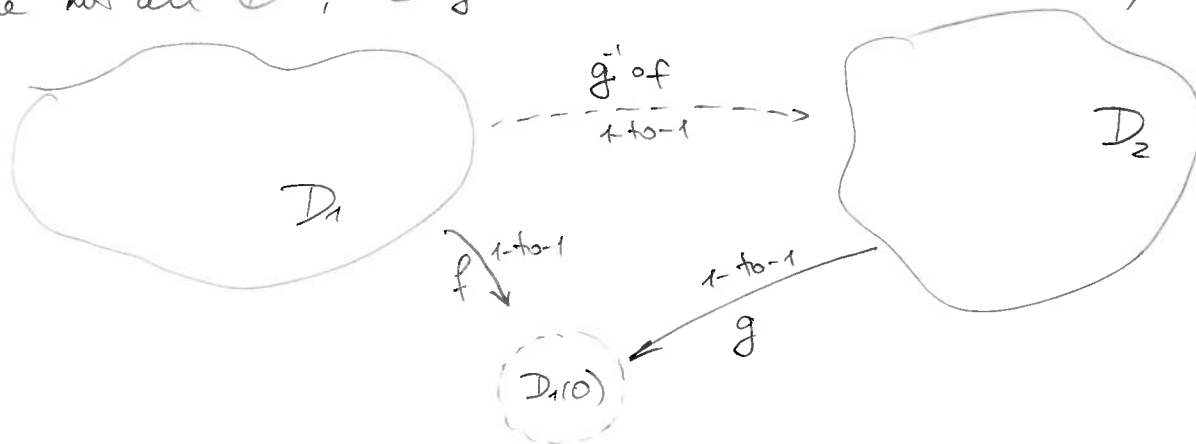
Theorem 6 (Riemann Mapping Theorem):

- a) Let D be any simply connected domain in $\mathbb{C}$, but not entire $\mathbb{C}$.
Then there is 1-to-1 analytic function f mapping D onto $\underline{D_1(0)}$
unit disk
- b) For any $z_0 \in D$ and direction vector through z_0 , the map f in a)
is unique such that $f(z_0) = 0 \in D_1(0)$ and f maps direction vector
to the direction of positive real axis

↑ we shall not prove this harder result.

[Note: a) f is quite implicit (no construction for it)
b) data in b)-part gives "3 degrees of freedom".

In particular, given two simply connected domains D_1, D_2 in $\mathbb{C}$
which are not all $\mathbb{C}$, we get that there is 1-to-1 analytic map
Btw them:



Next lectures: some explicit examples of such f .