Lecture #37

• Last time: Linear transformations & $\underbrace{\text{Linear Fractional}}_{=\text{Möbius}}$ Transformations

Ex1 (True / False): a) Composition of linear transformations is linear.

b) Inverse of linear transformation is linear.

▸ a) If $L_1(z) = az+b$ with $a \in \mathbb{C}\setminus 0$, $b \in \mathbb{C}$
$L_2(z) = \tilde{a}z + \tilde{b}$ with $\tilde{a} \in \mathbb{C}\setminus 0$, $\tilde{b} \in \mathbb{C}$ $\Bigg\} \Rightarrow (L_1 \circ L_2)(z) = a(\tilde{a}z+\tilde{b})+b$
$= \underbrace{a\tilde{a}}_{\neq 0} \cdot z + (a\tilde{b}+b)$

$\Rightarrow L_1 \circ L_2$ is also linear $\Rightarrow$ True!

b) If $w = L_1(z) = az+b$, then $z = \frac{w-b}{a} = a^{-1}\cdot w - a^{-1}\cdot b \Rightarrow L_1^{-1}(w) = \underbrace{a^{-1}}_{\neq 0}\cdot w - a^{-1}\cdot b$

$\Rightarrow$ inverse L_1^{-1} is also linear $\Rightarrow$ True!

Ex 2 (True / False): Composition of any sequence of translation, magnification, and rotation is always linear.

▸ As each of above three transformations is linear, the previous exercise shows that so is their composition $\Rightarrow$ True!

Rmk: Last time we verified that every linear transformation is a composition of above three basic ones, but now we see that the converse is also true.

Ex3 (True / False): a) Composition of Möbius transf. is again Möbius

b) Inverse of any Möbius transformation is again Möbius

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- (Solution of Ex3)

a) Let $L_1(z) = \dfrac{az+b}{cz+d}$, $L_2(z) = \dfrac{\tilde{a}z+\tilde{b}}{\tilde{c}z+\tilde{d}}$ with $ad-bc \neq 0$, $\tilde{a}\tilde{d}-\tilde{b}\tilde{c} \neq 0$. Then

$$(L_1 \circ L_2)(z) = \frac{a \cdot \frac{\tilde{a}z+\tilde{b}}{\tilde{c}z+\tilde{d}} + b}{c \cdot \frac{\tilde{a}z+\tilde{b}}{\tilde{c}z+\tilde{d}} + d} = \frac{(a\tilde{a}+b\tilde{c})z + (a\tilde{b}+b\tilde{d})}{(c\tilde{a}+d\tilde{c})z + (c\tilde{b}+d\tilde{d})}$$

Also: $(a\tilde{a}+b\tilde{c})(c\tilde{b}+d\tilde{d}) - (c\tilde{a}+d\tilde{c})(a\tilde{b}+b\tilde{d})$

$$= \cancel{ac\cdot\tilde{a}\tilde{b}} + ad\cdot\tilde{a}\tilde{d} + bc\cdot\tilde{b}\tilde{c} + \cancel{bd\cdot\tilde{c}\tilde{d}} - \cancel{ac\cdot\tilde{a}\tilde{b}} - bc\cdot\tilde{a}\tilde{d} - ad\cdot\tilde{b}\tilde{c} - \cancel{bd\cdot\tilde{c}\tilde{d}}$$

$$= (ad-bc)(\tilde{a}\tilde{d}-\tilde{b}\tilde{c}) \neq 0$$

So: $L_1 \circ L_2$ is again a Möbius transformation $\Rightarrow$ True!

b) If $w = L_1(z) = \dfrac{az+b}{cz+d}$, then $az+b = cwz+dw \Rightarrow z = \dfrac{dw-b}{-cw+a}$

$\Rightarrow$ inverse transformation L_1^{-1} is $L_1^{-1}(w) = \dfrac{dw-b}{-cw+a}$ with $da-bc \neq 0$

$\Rightarrow$ again Möbius transformation $\Rightarrow$ True!

Ex4 (True/False): Composition of any sequence of translation, magnification, rotation, and inversion is always a Möbius transform.

- As each of these basic 4 transformations is Möbius, so is their arbitrary composition by Ex3 $\Rightarrow$ True!

Rmk: Last time we verified that every Möbius transformation is a composition of above 4 basic ones, but now we see that the converse is also true.

Note:
- $\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \dfrac{1}{ad-bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$ and this is compatible with above f-la $L_1^{-1}(z)$
- $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot \begin{pmatrix} \tilde{a} & \tilde{b} \\ \tilde{c} & \tilde{d} \end{pmatrix} = \begin{pmatrix} a\tilde{a}+b\tilde{c} & a\tilde{b}+b\tilde{d} \\ c\tilde{a}+d\tilde{c} & c\tilde{b}+d\tilde{d} \end{pmatrix}$ which is compatible with above f-la for $L_1 \circ L_2$.

(in fact, the group of Möbius transformations is $PGL_2(\mathbb{C})$)

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Let's now solve the exercise from the very end of Lecture #36:

Ex 5: Find a conformal map of the unit disk $D_1(0) = \{z \in \mathbb{C} : |z| < 1\}$ onto the upper half plane $\mathcal{H} = \{w \in \mathbb{C} : \operatorname{Im} w > 0\}$

- Idea: Look for a Möbius transformation f mapping the unit circle $C_1(0) = \{z : |z| = 1\}$ onto the real line $\mathbb{R} = \{w \in \mathbb{C} : \operatorname{Im} w = 0\}$ and then make sure that interior of $C_1(0)$ is mapped to $\mathcal{H}$, not $-\mathcal{H}$.

- Note: $f(C_1(0))$ - line $\iff f$ has a pole on $C_1(0)$. Thus, there are two points $z_1, z_3 \in C_1(0)$ s.t. $f(z_1) = 0$, $f(z_3) = \infty$ (we use z_3 not z_2 for the latter discussion)

Let's make some choice, e.g. $z_1 = -1$, $z_3 = 1$. Then writing $f(z) = \frac{az+b}{cz+d}$ we see $a - b = 0 = c + d$, so that f can be written $f(z) = A \cdot \frac{z+1}{z-1}$ (with $A = \frac{a}{c} \in \mathbb{C} \setminus \{0\}$). For any $A \neq 0$, the image of $C_1(0)$ is thus a line passing through the origin.

- To make sure, this line $\mathbb{R}$, pick any other point on $C_1(0)$, e.g. $z_2 = i$. Then $f(i) = A \cdot \frac{i+1}{i-1} = A \cdot \frac{1}{i}$, hence $f(i) \in \mathbb{R} \iff A \in i \cdot \mathbb{R}$

- Above shows that for any $t \in \mathbb{R} \setminus 0$, the map $z \mapsto f(z) = it \cdot \frac{z+1}{z-1}$ maps $C_1(0)$ onto $\mathbb{R}$. But we also need interior to be mapped to $\mathcal{H}$ not $-\mathcal{H}$. To do so, pick any point inside, e.g. $z_4 = 0$. Then $f(0) = it \cdot \frac{1}{-1} = -it$ which is in $\mathcal{H}$ iff $t < 0$

So: A possible answer is $f(z) = -i \cdot \frac{z+1}{z-1} = i \cdot \frac{1+z}{1-z}$ ∎

An importance of such maps is illustrated by the following exercise:

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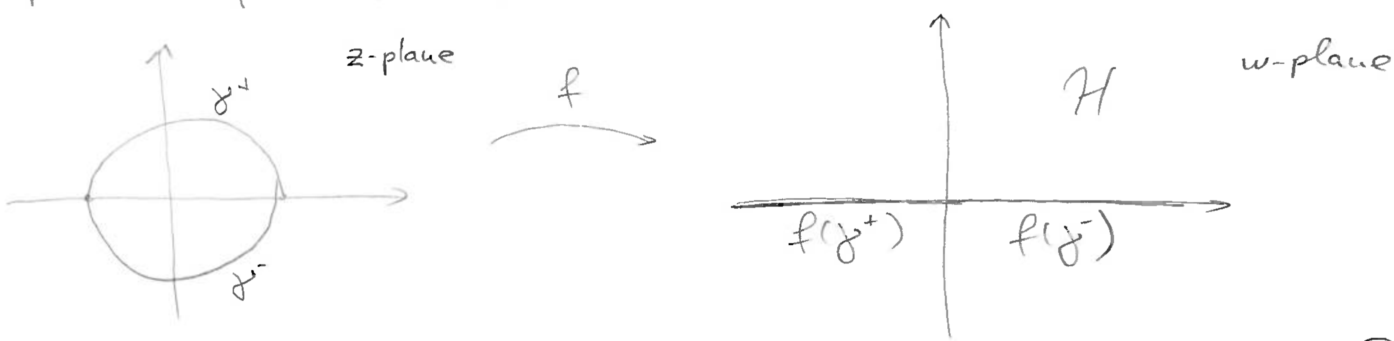
Ex 6 ($\sim$[Example 1 in Section 7.1]): Find a harmonic function $\phi(x,y)$ on the open unit disk $D_1(0)$ such that

$$\phi(x,y) \to +1 \text{ in } \mathcal{H} \cap C_1(0) = \text{upper half-circle} =: \gamma^+$$
$$\phi(x,y) \to -1 \text{ in } (-\mathcal{H}) \cap C_1(0) = \text{lower half-circle} =: \gamma^-$$

▶ Consider $f = i \cdot \frac{1+z}{1-z}$ from Ex 5 which maps $D_1(0) \xrightarrow[1:1]{\text{onto}} \mathcal{H}$

As $f(-1)=0$, $f(i)=-1$, $f(1)=\infty$, we see that $f(\gamma^+) = \mathbb{R}_{\leq 0}$

$f(-1)=0$, $f(-i)=\frac{i}{i}=1$, $f(1)=\infty$, we get $f(\gamma^-) = \mathbb{R}_{\geq 0}$



Pick a harmonic $\tilde{\phi}$ on $\mathcal{H}$ with boundary condition $\tilde{\phi} \to -1$ in $\mathbb{R}_{>0}$, $\tilde{\phi} \to +1$ in $\mathbb{R}_{<0}$

by solving a wedge problem (see Lecture 11). For example, we look for $\tilde{\phi} = A \cdot \arg_{-\pi/2} w + B$ so that $\left.\begin{array}{l} B=-1 \\ A\cdot\pi + B = 1 \end{array}\right\} \to \left\{\begin{array}{l} A = 2/\pi \\ B = -1 \end{array}\right.$

$\Rightarrow$ can take $\tilde{\phi} = \frac{2}{\pi} \arg_{-\pi/2} w - 1$

$\Rightarrow$ $\phi := \tilde{\phi} \circ f$ solves the problem, so that

$$\phi(x,y) = \frac{2}{\pi} \arg_{-\pi/2}\left(i \cdot \frac{1+z}{1-z}\right) - 1 = \frac{2}{\pi}\left(\arg_{-\pi/2}\left(\frac{1+z}{1-z}\right) + \frac{\pi}{2}\right) - 1 = \frac{2}{\pi} \arg_{-\pi/2} \frac{1+z}{1-z}$$

<u>Here</u>: $z = x+iy \Rightarrow \frac{1+z}{1-z} = \frac{1+x+iy}{1-x-iy} = \frac{(1+x+iy)(1-x+iy)}{(1-x)^2+y^2} = \frac{1-x^2-y^2+i\cdot 2y}{(1-x)^2+y^2}$

$$\Rightarrow \phi(x,y) = \frac{2}{\pi} \tan^{-1} \frac{2y}{1-x^2-y^2}$$

[Rem: This solution is much easier than applying Poisson integral formula (see Lecture 18).

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As we saw in Solution of Ex5, the key was to find a Möbius map f so that $f: C_1(0) \xrightarrow[1:1]{\text{onto}} \mathbb{R}$. This admits the following natural generalization:

Question 1: Find a Möbius map f which maps any given "generalized circle" C_1 onto another given "generalized circle" C_2.

Here, a "generalized circle" means either a circle or a line (by Theorem from Lecture 36, we know Möbius maps : gen. circle $\to$ gen. circle). The latter is determined by any 3 points on it (we get a line if the points are on a line).

Hence, the above question admits the following upgrade:

Question 2: Given any three distinct points z_1, z_2, z_3 in z-plane, and any three distinct points w_1, w_2, w_3 in w-plane, find a Möbius map f s.t. $f(z_1)=w_1$, $f(z_2)=w_2$, $f(z_3)=w_3$

To answer this, let's first find a Möbius map L_1 so that $L_1(z_1)=0$, $L_1(z_2)=1$, $L_1(z_3)=\infty$. Arguing as in the sol-n of Ex5:

$L_1(z) = A \cdot \frac{z-z_1}{z-z_3}$ for some $A \in \mathbb{C}\setminus\{0\}$ (which encodes $L_1(z_1)=0$, $L_1(z_3)=\infty$)

Plug $z=z_2$ to get $1 = L_1(z_2) = A \cdot \frac{z_2-z_1}{z_2-z_3} \Rightarrow A = \frac{z_2-z_3}{z_2-z_1}$.

So: $\boxed{L_1(z) = \frac{z-z_1}{z-z_3} \cdot \frac{z_2-z_3}{z_2-z_1}}$ $\leftarrow$ This answers Question 2 in the special case $w_1=0$, $w_2=1$, $w_3=\infty$.

But we can likewise consider Möbius map L_2 so that $L_2(w_1)=0$, $L_2(w_2)=1$, $L_2(w_3)=\infty$ which is given by formula alike:

$$\boxed{L_2(w) = \frac{w-w_1}{w-w_3} \cdot \frac{w_2-w_3}{w_2-w_1}}$$

But then the composition $L_2^{-1} \circ L_1 =: f$ is a Möbius transformation (by Ex3) and satisfies $f(z_1)=w_1$, $f(z_2)=w_2$, $f(z_3)=w_3$, which answers above Question 2!

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In fact, we claim that a solution f to Question 2 is unique:

Lemma 1: If S, T are Möbius transformations such that $S(z) = T(z)$ at 3 points $z \in \{z_1, z_2, z_3\}$, then in fact $S = T$!

▸ Consider $L := S^{-1} \circ T$ which is also a Möbius transformation and satisfies $L(z) = z$ at three distinct points. Then, by the next lemma, $L(z) \equiv z$ for all z $\Rightarrow S = T$ ∎

Lemma 2: Any Möbius map that fixes three distinct points is identity

▸ Let $L(z) = \dfrac{az+b}{cz+d}$.

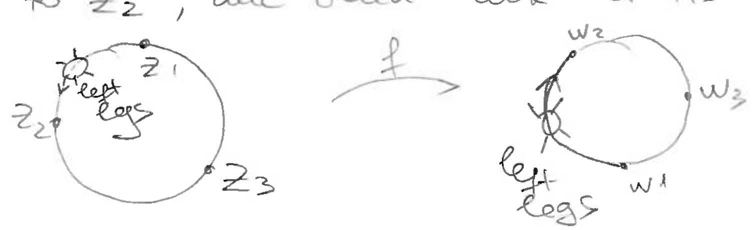
Case 1 ($c=0$): $\dfrac{az+b}{d} = z \Leftrightarrow (a-d)z + b = 0$ has 3 sol-s only if $\begin{cases} b=0 \\ a=d \end{cases}$

$\Rightarrow L(z) = \dfrac{az}{a} = z \Rightarrow L = \text{identity}$

Case 2 ($c \neq 0$): $\dfrac{az+b}{cz+d} = z \Leftrightarrow c \cdot z^2 + (d-a)z - b = 0$ has 3 sol-s only if $\begin{cases} c=0 \\ b=0 \\ a=d \end{cases}$

$\Rightarrow$ contradiction with $c \neq 0$ ∎

Remark: "Bug test" Now that we know how to construct a Möbius transformation f mapping a circle C_1 to a circle C_2, so that $z_1, z_2, z_3 \in C_1$ are mapped to $w_1, w_2, w_3 \in C_2$, respectively, it still remains to understand if f maps interior of C_1 to interior or exterior of C_2 (if C_1 or C_2 is a line, then we get two half planes). To this end, imagine you have a "bug" crawling from z_1 to z_2, and then look at its "image" from w_1 to w_2



In this picture $f(\text{interior}) = \text{exterior}$