Lecture #37 - supplemental material

We ended Lecture #37 with a statement of two important results which are not on any of exams, but are important to know in the context of Möbius transformations and Riemann Mapping Thm. While the proofs were omitted in class - we present them below.

Theorem 1 ("Schwarz lemma"): Let f be an analytic map $f: D_1(0) \circlearrowleft$ (open unit disk) such that $f(0)=0$. Then:

A) $|f(z)| \leq |z|$ for any $z \in D_1(0)$

B) $|f'(0)| \leq 1$.

Moreover, if equality holds in A) for some $z \neq 0$, or holds in B), then in fact $f(z) \equiv \lambda \cdot z$ with $|\lambda| = 1$, i.e. f is a rotation of $D_1(0)$

▶ Consider $F(z) = \frac{f(z)}{z}$ for any $z \in D_1(0) \setminus \{0\}$. Note that $\lim_{z \to 0} F(z) = \lim_{z \to 0} \frac{f(z) - f(0)}{z - 0} = f'(0)$ $\Rightarrow$

$\Rightarrow$ F has a removable singularity at $z_0 = 0$ and redefining it at $z=0$ via $F(0) = f'(0)$ gives an analytic function on $D_1(0)$.

Now: For any $z_1 \in D_1(0)$ pick any real $0 < \rho < 1$ so that $|z_1| < \rho$. Then by the "Maximal modulus principle" (Lecture 17), applied to function $F(z)$ on the disk $\overline{D_\rho(0)} = \{z \in \mathbb{C} : |z| \leq \rho\}$, we get:

$$|F(z_1)| \leq \max_{z \in C_\rho(0)} |F(z)| = \frac{1}{\rho} \max_{z \in C_\rho(0)} |f(z)| \leq \frac{1}{\rho}$$

In the limit $\rho \to 1$, we thus get $|F(z_1)| \leq 1$ for all $z_1 \in D_1(0)$

For $z_1 \neq 0$, this gives A), while for $z_1 = 0$, this gives B)

Finally, if equality holds, then by the same "Max modulus principle" $F(z) \equiv \lambda$ - constant on $D_1(0)$ and $|\lambda| = 1$, which establishes last part ☐

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With the help of Schwarz lemma, we can now establish:

Theorem 2: The group of automorphisms of the unit disk $D_1(0)$, i.e.

$$Aut(D_1(0)) = \{\text{all analytic onto 1-to-1 maps } f: D_1(0) \circlearrowleft\}$$

consists of the following Möbius transformations

$$\left\{\lambda \cdot \frac{z-a}{1-\bar{a}z} \;\middle|\; a \in D_1(0) \text{ and } |\lambda| = 1\right\}$$

Step 1: First, let's verify that every Möbius transformation

$$z \mapsto \lambda \cdot \frac{z-a}{1-\bar{a}z} \quad \text{with } |\lambda|=1,\ a \in D_1(0)$$

defines an automorphism of $D_1(0)$. As $\lambda \cdot (\dots)$ is just a rotation around 0 (since $|\lambda|=1$), we can assume $\lambda=1$ now.

Let $\varphi_a(z) = \frac{z-a}{1-\bar{a}z}$. Then, for any point $z = e^{i\theta} \in C_1(0)$, we get:

$$\varphi_a(e^{i\theta}) = \frac{e^{i\theta}-a}{1-\bar{a}\cdot e^{i\theta}} = \frac{(e^{i\theta}-a)e^{-i\theta}}{(1-\bar{a}e^{i\theta})e^{-i\theta}} = \underbrace{e^{-i\theta}}_{\text{has abs. value}=1} \cdot \underbrace{\frac{e^{i\theta}-a}{e^{-i\theta}-\bar{a}}}_{\text{has abs. value}=1}$$

has abs. value $=1$ as denominator is cpx conjugate of numerator.

So: $\varphi_a : C_1(0) \to C_1(0)$. Also $\varphi_a(a) = 0 \Rightarrow$ interior $\mapsto$ interior (as $|a|<1$)

This shows that $\varphi_a \in Aut(D_1(0))$.

Step 2: Note that the inverse to φ_a is given by (see Ex 3):

$$\varphi_a^{-1}(z) = \frac{z+a}{\bar{a}z+1} = \varphi_{-a}(z)$$

As $\varphi_a : a \mapsto 0$, we see that $\varphi_{-a} = \varphi_a^{-1} : 0 \mapsto a$.

Step 3: Let's now pick any $f \in Aut(D_1(0))$. Our goal is to show that $f = \lambda\varphi_a$ for some $|\lambda|=1$, $|a|<1$.

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(Continuation of Proof)

Let $a = f^{-1}(0) \in D_1(0)$. Consider then the composition

$$g := f \circ \varphi_a^{-1}$$

Then: g is an analytic map $D_1(0) \circlearrowleft$ with $g(0) = 0$.

Hence by Schwarz lemma (Theorem 1 above): $|g'(0)| \leq 1$.

But: g is an automorphism of $D_1(0)$ and so:

$h := g^{-1}$ is also an analytic map $D_1(0) \circlearrowleft$ with $h(0) = 0$

Hence by Schwarz lemma: $|h'(0)| \leq 1$

However: $h'(0) = \frac{1}{g'(0)}$ by the formula for derivative of inverse.

Combining this with $|g'(0)| \leq 1$, $|h'(0)| \leq 1$, we get $|g'(0)| = 1$.

But by the last part of Schwarz lemma $\Rightarrow g(z) \equiv \lambda \cdot z$

for some $\lambda \in \mathbb{C}$ with $|\lambda| = 1$

As $g = f \circ \varphi_a^{-1}$, we finally get $f(z) = \lambda \cdot \varphi_a(z)$ as claimed $\blacksquare$

Remark: We note that Theorem 2 is a special case of the Riemann Mapping Theorem (= Theorem 6 of Lecture 35) for $D = D_1(0)$, where in part b), $a = z_0$, while λ is determined by the condition that f maps a given direction vector through z_0 to the direction of positive real axis