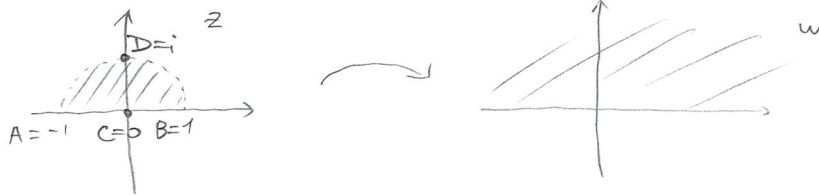


Lecture #40

1

Ex1: Find a conformal map that maps $D = \{z: |z| < 1 \text{ \& } \text{Im } z > 0\}$ onto $H = \{w: \text{Im } w > 0\}$

Want:



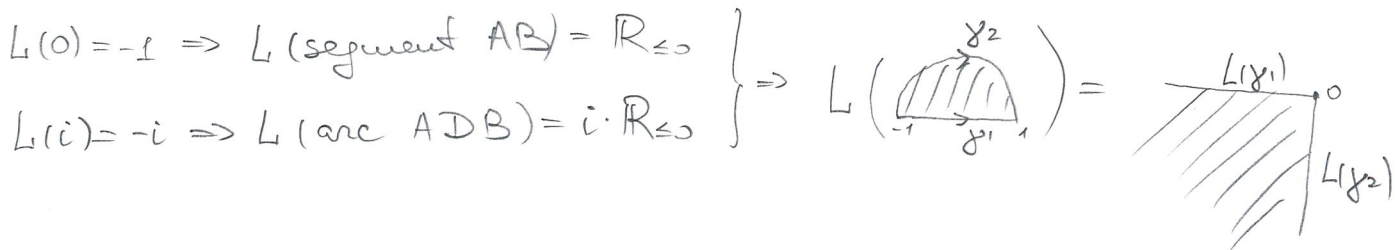
Step 1: Use a Möbius map L that will map D onto some wedge

E.g. if $A \mapsto 0, B \mapsto \infty$, then $L = \text{const} \cdot \frac{z+1}{z-1}$. Consider such map with $\text{const}=1$.

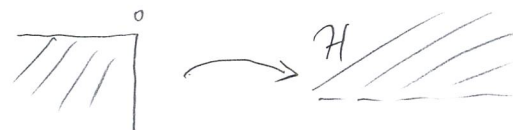
$$L(z) = \frac{z+1}{z-1}$$

Note: $L(0) = -1 \Rightarrow L(\text{segment } AB) = \mathbb{R}_{\leq 0}$

$L(i) = -i \Rightarrow L(\text{arc } ADB) = i \cdot \mathbb{R}_{\leq 0}$

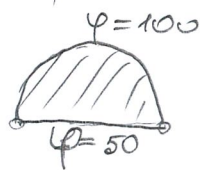


Step 2: Apply $z \mapsto z^2 = (z)^2$ to map above wedge

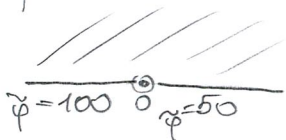


So: One of possible answers is $f(z) = \left(\frac{z+1}{z-1}\right)^2$

Ex2: Solve Dirichlet problem on $D_1(0) \cap H$ with boundary conditions



Apply $f(z) = \left(\frac{z+1}{z-1}\right)^2$ from Ex1 to map above onto H . Then boundary conditions read

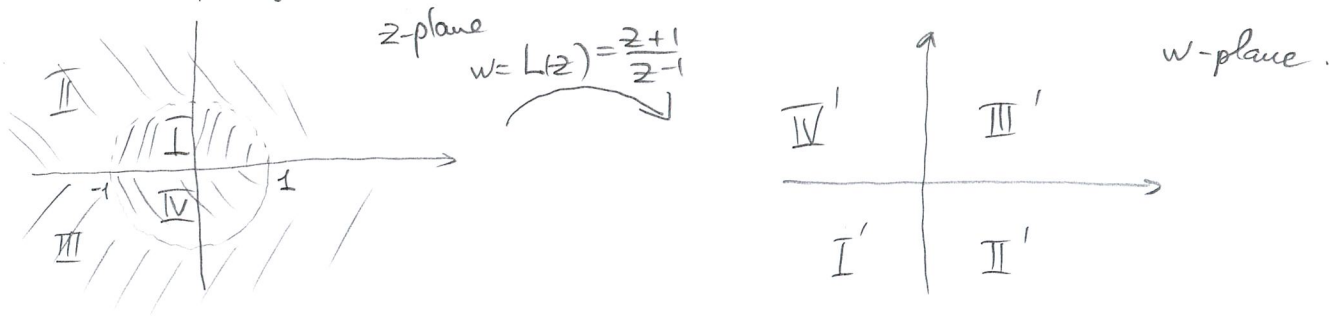


Look for harmonic $\varphi(w) = A \cdot \arg_{-\pi/2} w + B$. Then $B = 50, A \cdot \pi + 50 = 100 \Rightarrow A = \frac{50}{\pi}$

So: $\varphi(z) = \frac{50}{\pi} \arg_{-\pi/2} \left(\left(\frac{z+1}{z-1}\right)^2\right) + 50$ is an example.

Lecture #40

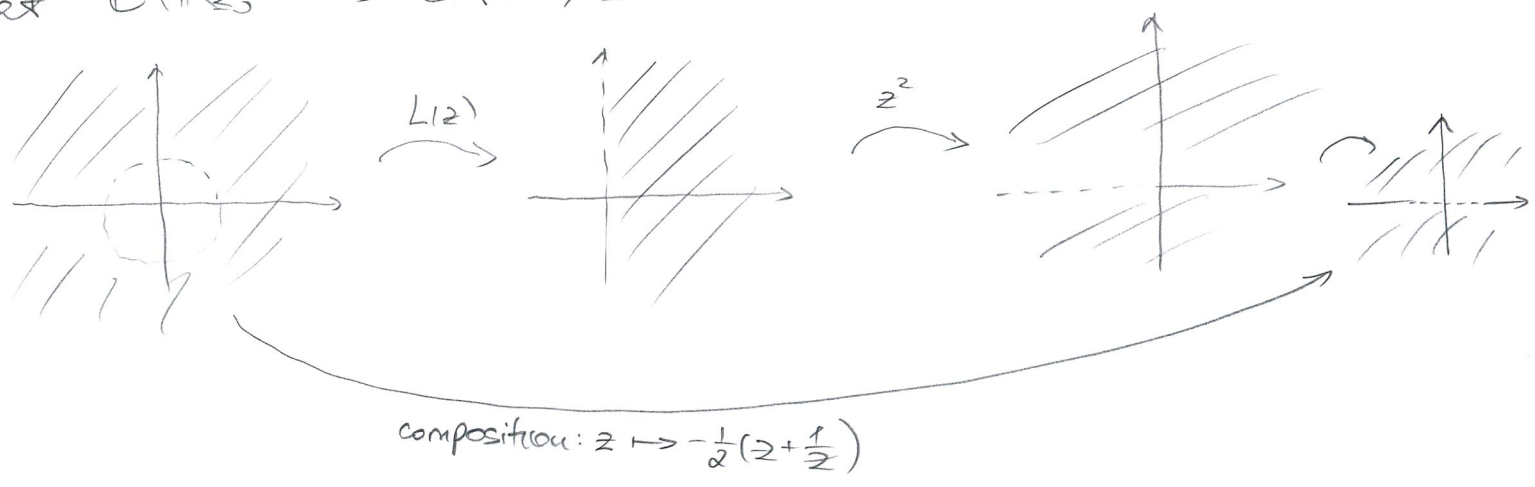
Back to Step 1 from selection of Ex1 we note that



In particular, the exterior of unit circle, i.e. $II \cup III$ are mapped to the right half plane $\text{Re } w > 0$. Applying further $z \mapsto z^2$, we get



Applying further the Möbius map $z \mapsto \frac{z+1}{-z+1}$ s.t. $0 \mapsto 1$, $\infty \mapsto -1$, we see that $\mathbb{C} \setminus \mathbb{R}_{\leq 0} \rightarrow \mathbb{C} \setminus [-1, 1]$

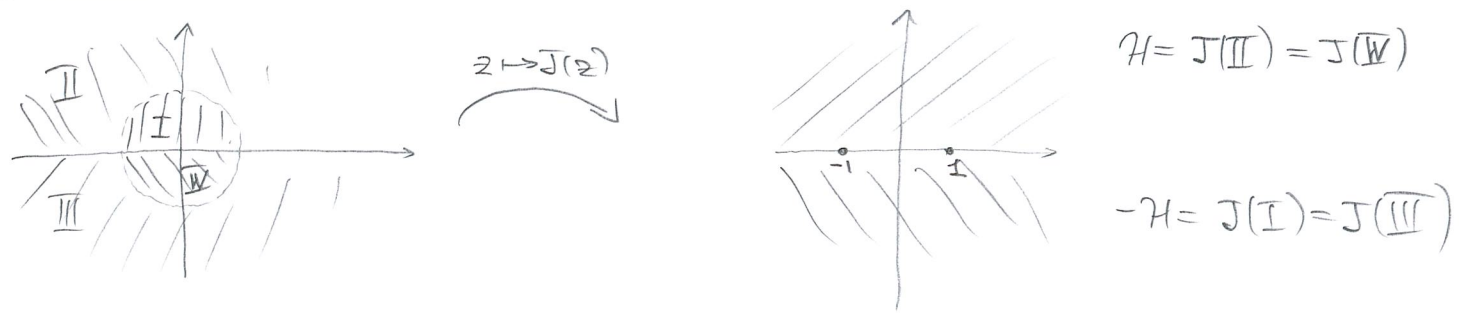


Def: $z \mapsto J(z) = \frac{1}{2}(z + \frac{1}{z})$ is a (famous) Jukovsky map.

In polar coordinates, if $z = re^{i\theta}$ then $J(z = re^{i\theta}) = \frac{r+z^{-1}}{2} \cos \theta + i \cdot \frac{r-z^{-1}}{2} \sin \theta$

Note: Image of any circle $|z|=r$ is now an ellipse $\left(\frac{x}{\frac{r+z^{-1}}{2}}\right)^2 + \left(\frac{y}{\frac{r-z^{-1}}{2}}\right)^2 = 1$ except for unit circle $|z|=1$ which is mapped to the segment $[-1, 1]$

Lecture #40



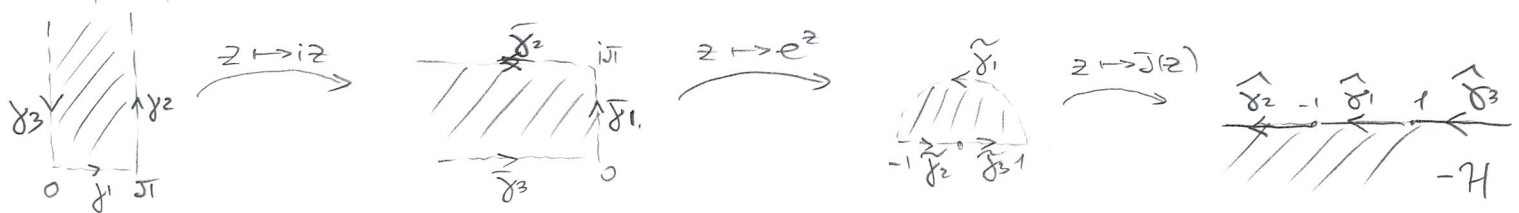
Note: $[1, +\infty)$ is the image of $(0, 1]$.
 $(-\infty, -1]$ is the image of $[-1, 0)$
 $[-1, 1]$ is the image of both top and bottom halves of unit circle

Also: $J(z) = J(1/z)$ and the map $z \mapsto 1/z$ maps $\text{II} \leftrightarrow \text{IV}$, $\text{I} \leftrightarrow \text{III}$.

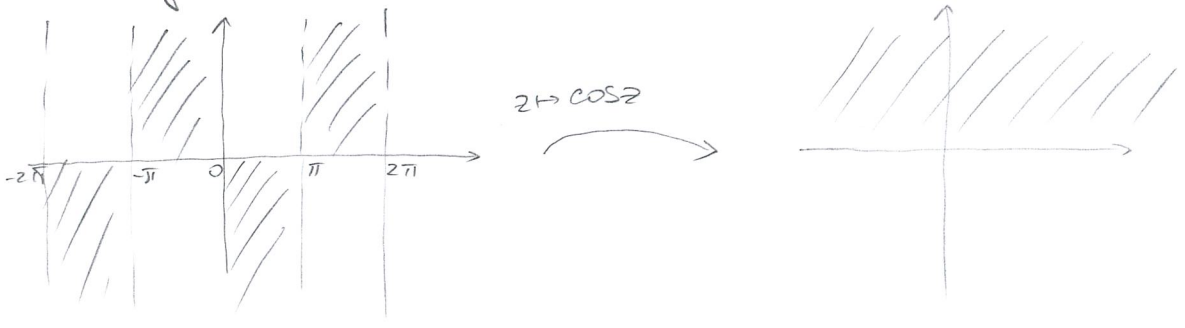
Recall that $\cos z = \frac{e^{iz} + e^{-iz}}{2}$, so that

$\cos(z) = J(e^{iz})$

Let's now visualize the geometry of the cosine map. We start with a half of vertical strip $\{z \in \mathbb{C} : \text{Im} z > 0, 0 \leq \text{Re} z < \pi\}$



Combining this with $\cos(-z) = \cos z$ and $\cos(z + \pi) = -\cos z$, we get

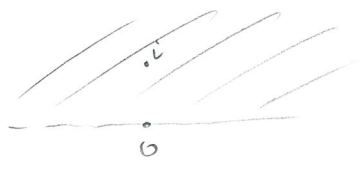


Combining this with $\sin z = -\cos(z + \pi/2)$ one also gets geometry of sh-map

In particular $\{z \in \mathbb{C} : -\pi/2 \leq \text{Re} z < \pi/2\}$ maps to $\mathcal{H}$.

Lecture #40

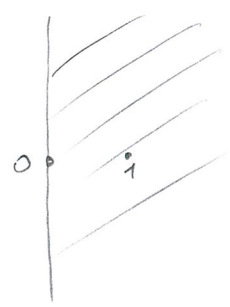
Let's just conclude with some useful toolbox of conformal maps



$$L = \frac{z-i}{z+i}$$

$$L^{-1}(w) = \frac{iw+i}{-w+i}$$

$L(0) = -1$ $0 = L(i)$ $L(\infty) = 1$
 $L(1) = -i$



$$L = \frac{z-1}{z+1}$$

$$L^{-1} = \frac{w+1}{-w+1}$$

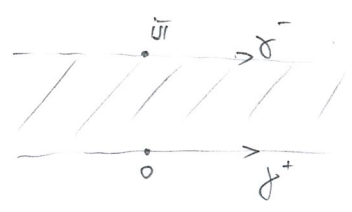
$L(i) = i$
 $L(0) = -1$ $0 = L(1)$ $L(\infty) = 1$



$$z \mapsto z^{\pi/\alpha}$$

$$w \mapsto w^{\alpha/\pi}$$

$e^{\frac{\pi}{\alpha} \log z}$



$$z \mapsto e^z$$

e^{y+} e^{y-}



$$\cos$$

$-H$



$$\sin$$

H