Lecture #26

Last time we finished our discussion of the non-degenerate pairing
$$(\cdot,\cdot): U_q^{\leq} \times U_q^{\geq} \longrightarrow \Bbbk.$$

<u>Today</u>: A bilinear pairing $\langle\cdot,\cdot\rangle: U_q(\mathfrak{g}) \times U_q(\mathfrak{g}) \to \Bbbk$. To define it, recall the triangular decomposition mult: $U_q^- \otimes U_q^0 \otimes U_q^+ \xrightarrow{\sim} U_q(\mathfrak{g})$. Define:

$$\boxed{\begin{array}{c}\langle (yK_\nu)\cdot K_\lambda \cdot x, (y'K_{\nu'})K_{\lambda'}\cdot x'\rangle := (y',x)\cdot(y,x')\cdot q^{(2\rho,\nu)}\cdot (q^{+1/2})^{-(\lambda,\lambda')} \\ \text{for any } \nu,\nu',\mu,\mu',\lambda,\lambda' \in Q \text{ and} \\ x\in (U_q^+)_\mu,\ x'\in(U_q^+)_{\mu'},\ y\in (U_q^-)_{-\nu},\ y'\in(U_q^-)_{-\nu'}\end{array}}$$

<u>Note</u>: The reason to split off K_ν and $K_{\nu'}$ from K_λ and $K_{\lambda'}$ is to make the q-power in the right-hand side simple enough.

$$\boxed{\underline{\text{Proposition 1}}:\ \langle \mathrm{ad}(u)v, v'\rangle = \langle v, \mathrm{ad}(S(u))v'\rangle \quad \forall u,v,v'\in U_q(\mathfrak{g})}$$

<u>Exercise</u>: Verify that the equality above is equivalent to
$U_q(\mathfrak{g})\otimes U_q(\mathfrak{g}) \xrightarrow{\langle,\rangle} \Bbbk$ being a $U_q(\mathfrak{g})$-module morphism.

▸ By linearity, we may assume that
$v = (yK_\nu)K_\lambda x$, $v' = (y'K_{\nu'})K_{\lambda'}x'$ as above.

Clearly, it suffices to verify the equality for $u \in \{K_i, E_i, F_i\}_{i\in I}$.

• <u>$u = K_i$</u>

In this case:

$\mathrm{ad}(u)v = K_i v K_i^{-1} = v\cdot q^{(\alpha_i,\mu-\nu)}$

$\mathrm{ad}(S(u))v' = K_i^{-1}v'K_i = v'\cdot q^{-(\alpha_i,\mu'-\nu')}$

If $\mu\neq\nu'$ or $\mu'\neq\nu$, then both sides vanish. But if $\mu=\nu'$, $\mu'=\nu$, then $(\alpha_i,\mu-\nu) = -(\alpha_i,\mu'-\nu')$ and the equality follows.

Lecture #26

• $u = E_i$

By [Lecture #25, Lemma 5]:

$$\boxed{\begin{aligned} ad(E_i)v = {} & q^{-(\nu,\alpha_i)}\cdot(yK_\nu)\cdot K_\lambda\cdot\left(q^{-(\lambda,\alpha_i)}E_i x - q^{(\mu,\alpha_i)}xE_i\right) + \\ & + \frac{1}{q_i-q_i^{-1}}\cdot\left(q^{-(\nu-\alpha_i,\alpha_i)}\left(r_i(y)K_{\nu-\alpha_i}\right) - r_i'(y)K_{\nu-\alpha_i}\right)x \end{aligned}}$$

Also note that $S(E_i) = -K_i^{-1}E_i \Rightarrow ad(S(E_i)) = -ad(K_i^{-1})ad(E_i)$.

Exercise (direct application of same [Lecture 25, Lemma 5]):

$$\boxed{\begin{aligned} ad(S(E_i))v' = {} & -q^{-(\alpha_i,\alpha_i)}\cdot(y'K_{\nu'})K_{\lambda'}\left(q^{-(\mu'+\lambda',\alpha_i)}E_i x' - x'E_i\right) \\ & - \frac{q^{-(\mu',\alpha_i)}}{q_i-q_i^{-1}}\left(\left(r_i(y')K_{\nu'-\alpha_i}\right)K_{\lambda'+2\alpha_i} - q^{(\nu'-\alpha_i,\alpha_i)}\left(r_i'(y')K_{\nu'-\alpha_i}\right)K_{\lambda'}\right)x' \end{aligned}}$$

Note: $\left.\begin{array}{l} xE_i,\ E_i x \in (U_q^+)_{\mu+\alpha_i} \\ r_i(y),\ r_i'(y) \in (U_q^-)_{-\nu+\alpha_i} \end{array}\right\} \Rightarrow$ both sides in the desired equality vanish unless:

(I) $\nu' = \mu+\alpha_i$, $\nu = \mu'$

or

(II) $\nu' = \mu$, $\nu = \mu'+\alpha_i$.

Case (I)

$$\langle ad(E_i)v, v'\rangle = q^{-(\nu,\alpha_i)}(y,x')\,q^{(2\rho,\nu)}(q^{1/2})^{-(\lambda,\lambda')}\cdot\left(q^{-(\lambda,\alpha_i)}(y',E_i x) - q^{(\mu,\alpha_i)}(y',xE_i)\right)$$

$$\langle v, ad(S(E_i))v'\rangle = -\frac{1}{q_i-q_i^{-1}}\,q^{-(\mu',\alpha_i)}\,q^{(2\rho,\nu)}(y,x')\cdot\left(\begin{array}{l}(r_i(y'),x)(q^{1/2})^{-(\lambda,\lambda'+2\alpha_i)} - \\ -q^{(\nu'-\alpha_i,\alpha_i)}(r_i'(y'),x)(q^{1/2})^{-(\lambda,\lambda')}\end{array}\right)$$

Since $\nu = \mu'$, $\nu'-\alpha_i = \mu$, the above expressions coincide due to:

$$q^{-(\lambda,\alpha_i)}(y',E_i x) - q^{(\mu,\alpha_i)}(y',xE_i) = -\frac{q^{-(\lambda,\alpha_i)}}{q_i-q_i^{-1}}(r_i(y'),x) + \frac{q^{(\mu,\alpha_i)}}{q_i-q_i^{-1}}(r_i'(y'),x)$$

follows from $(y',E_i x) = \frac{-1}{q_i-q_i^{-1}}(r_i(y'),x)$, $(y',xE_i) = \frac{-1}{q_i-q_i^{-1}}(r_i'(y'),x)$

lied in Lecture 25.

: Treat Case (II) likewise

Proposition 2: Assuming the "tQ condition", $\langle , \rangle$ is non-degenerate.

▸ We shall prove non-deg. in the 2nd argument (the 1st - similar).
For degree reasons, assume that $u \in (U_q^-)_{-\nu} \cdot U_q^0 \cdot (U_q^+)_\mu$ is in 2nd kernel, i.e. $\langle v, u\rangle = 0$ $\forall v$.

Choose a basis $\{x_i^\mu\}_{i=1}^{N_\mu = \dim (U_q^+)_\mu}$ of $(U_q^+)_\mu$ and let $\{y_i^\mu\}_{i=1}^{N_\mu}$ be the dual basis of $(U_q^-)_{-\mu}$ w.r.t. $(\cdot,\cdot)$, which we proved to be nondeg.

Then: $\{(y_i^\nu K_\nu)\cdot K_\lambda \cdot x_j^\mu \mid 1\le i\le N_\nu, 1\le j\le N_\mu, \lambda\in Q\}$ - basis of $(U_q^-)_{-\nu} \cdot U_q^0 \cdot (U_q^+)_\mu$.

Moreover: $\langle (y_j^\mu K_\mu) K_{\lambda'} x_{i'}^{\nu'}, (y_i^\nu K_\nu) K_\lambda x_j^\mu\rangle = \delta_{ii'}\delta_{jj'} q^{(2\rho,\nu)} (q^{1/2})^{-(\lambda,\lambda')}$

Thus, if $u = \sum_{i,j,\lambda} \underbrace{a_{ij\lambda}}_{\in \Bbbk} (y_i^\nu K_\nu) K_\lambda x_j^\mu$, then we get:

$$\boxed{\sum_\lambda a_{ij\lambda} q^{-\frac{1}{2}(\lambda,\lambda')} = 0 \quad \forall i,j,\lambda'}$$

If $q \ne \sqrt[?]{1}$, then $q^{(\cdot, -\frac{1}{2}\lambda)}$ are pairwise distinct characters $Q \to \Bbbk^*$ and hence are linearly independent ("Artin's lemma") $\Rightarrow a_{ij\lambda} = 0$ $\forall i,j,\lambda$

So: $u = 0$ ∎

Rmk: If not for the ∞-dim Cartan piece, the above proof is elementary.

Lemma 1: Given any bilinear map $\varphi : (U_q^-)_{-\mu} \times (U_q^+)_\nu \to \Bbbk$ and $\lambda \in Q$ there exists $v \in (U_q^-)_{-\nu} K_{\lambda+\nu} (U_q^+)_\mu$ s.t.

$$\langle (yK_\mu) K_{\lambda'} x, v\rangle = \varphi(y,x)\cdot (q^{1/2})^{-(\lambda,\lambda')} \quad \forall x \in (U_q^+)_\nu, y \in (U_q^-)_{-\mu}, \lambda' \in Q$$

▸ Set $v := \sum_{i,j} \varphi(y_j^\mu, x_i^\nu)\, q^{(-2\rho,\nu)} \cdot (y_i^\nu K_\nu) K_\lambda x_j^\mu$. We claim it's as desired:

$\langle y_{j'}^\mu K_\mu K_{\lambda'} x_{i'}^\nu, v\rangle = \varphi(y_{j'}^\mu, x_{i'}^\nu)(q^{1/2})^{-(\lambda,\lambda')}$ ∎

Next time: Use the above lemma to prove quantum Harish-Chandra isom.