HOMEWORK 1 (DUE FEBRUARY 6)

1. (a) Verify that the 2-cocycle $\omega\colon W\otimes W\to\mathbb{C}$ defined via $\omega(L_n,L_m)=(n^3-n)\delta_{n,-m}$ is not a 2-coboundary (here W is the Witt algebra).

(b) For a finite dimensional Lie algebra $\mathfrak{g}$ with a nontrivial invariant symmetric bilinear form $(\cdot,\cdot)$, verify that the 2-cocycle $\omega\colon \mathfrak{g}[t,t^{-1}]\otimes\mathfrak{g}[t,t^{-1}]\to\mathbb{C}$ defined via $\omega(F,G)=\mathrm{Res}_{t=0}(dF,G)$ is not a 2-coboundary.

Note: This exercise completes our proofs of Theorems 1 from Lectures 1 and 2.

2. Consider the setup from Lecture 2: L is a Lie algebra, $\overline{L}\subset L$ is a Lie subalgebra, $M\subset L$ is an $\overline{L}$-submodule. Given a 2-cocycle $\omega\in Z^2(L)$, define $\varphi\colon\overline{L}\to M^*$ via $\varphi(x)(m):=\omega(x,m)$. Verify that $\varphi\in Z^1(\overline{L},M^*)$, i.e. φ is a 1-cocycle with values in M^*.

Note: This exercise was crucially used in the proof of Theorem 1 from Lecture 2.

3. Recall the Fock modules F_μ of the oscillator algebra $\mathcal{A}$ from Lecture 2.

(a) Construct an infinite-dimensional irreducible $\mathcal{A}$-representation, not isomorphic to any F_μ.

(b) For any $\mathcal{A}$-representation V, let $V[0]=\{v\in V\mid K(v)=v, a_0(v)=\mu v, a_n(v)=0\ \forall n>0\}$. Construct a natural $\mathcal{A}$-module homomorphism $F_\mu\otimes V[0]\to V$ and prove that it is injective.

4. (a) Consider a $\mathbb{Z}$-grading of $\mathfrak{g}=\mathfrak{sl}_n$ defined via $\deg(E_{ij})=j-i$. Verify that it makes $\mathfrak{g}$ into a non-degenerate $\mathbb{Z}$-graded Lie algebra.

(b) For $\mathfrak{g}=\mathfrak{sl}_n$, define a $\mathbb{Z}$-grading of $\widehat{\mathfrak{g}}=\mathfrak{g}[t,t^{-1}]\oplus\mathbb{C}K$ via $\deg(E_{ij}t^k)=j-i+nk, \deg K=0$. Verify if it makes $\widehat{\mathfrak{g}}$ and $\mathfrak{g}[t,t^{-1}]$ into non-degenerate $\mathbb{Z}$-graded Lie algebras.

Note: One can actually consider any simple finite dimensional $\mathfrak{g}$ with the *principal* grading.

5. A linear operator A on a $\mathbb{C}$-vector space V acts locally finitely if $\mathbb{C}[A]v$ is finite dimensional for any $v\in V$. Verify that if $W\subset V$ is an A-stable subspace (i.e. $A(W)\subset W$) and A acts locally finitely both on W and the quotient V/W, then it also acts locally finitely on V itself.

Note: This was used in the proof of Proposition 1 from Lecture 3.

6. For each of the Lie algebras from Lectures 1–2, with a $\mathbb{Z}$-grading introduced in Lecture 3, decide if there is an element $L\in\mathfrak{g}_0$ such that $[L,x]=\deg(x)\cdot x$ for any homogeneous $x\in\mathfrak{g}$.

Note: This condition was used in Theorem 1(c) from Lecture 4.

7. (a) Let $\mathfrak{a}$ be a Lie algebra, $\mathfrak{b}$ be a Lie subalgebra of $\mathfrak{a}$, M be a $\mathfrak{b}$-module, N be an $\mathfrak{a}$-module. Prove that

$$\mathrm{Ind}_{\mathfrak{b}}^{\mathfrak{a}}(M)\otimes N\simeq \mathrm{Ind}_{\mathfrak{b}}^{\mathfrak{a}}(M\otimes \mathrm{Res}_{\mathfrak{b}}^{\mathfrak{a}}(N))\quad\text{as}\quad \mathfrak{a}-\text{modules}.$$

(b) Let $\mathfrak{c}$ be a Lie algebra, $\mathfrak{a},\mathfrak{b}$ be two Lie subalgebras of $\mathfrak{c}$ such that $\mathfrak{a}+\mathfrak{b}=\mathfrak{c}$. Note that $\mathfrak{a}\cap\mathfrak{b}$ is also a Lie subalgebra of $\mathfrak{c}$. Let M be a $\mathfrak{b}$-module. Prove that

$$\mathrm{Res}_{\mathfrak{a}}^{\mathfrak{c}}(\mathrm{Ind}_{\mathfrak{b}}^{\mathfrak{c}}(M))\simeq \mathrm{Ind}_{\mathfrak{a}\cap\mathfrak{b}}^{\mathfrak{a}}(\mathrm{Res}_{\mathfrak{a}\cap\mathfrak{b}}^{\mathfrak{b}}(M))\quad\text{as}\quad \mathfrak{a}-\text{modules}.$$

Hint: You may wish to use $U(\mathfrak{a})\otimes_{U(\mathfrak{a}\cap\mathfrak{b})}U(\mathfrak{b})\simeq U(\mathfrak{c})$ both as left $\mathfrak{a}$ and right $\mathfrak{b}$ modules.

The last 3 problems are about representations of the Witt algebra W needed later in class.

8. (a) Show that W is a simple Lie algebra (i.e. it has no proper ideals).

(b) Deduce that W has no nontrivial finite dimensional representations.

9. For $\alpha, \beta \in \mathbb{C}$, let $V_{\alpha,\beta}$ be the vector space of formal expressions $g(t)t^\alpha(dt)^\beta$ with $g \in \mathbb{C}[t, t^{-1}]$ (*tensor fields* of rank β and branching α on the punctured complex plane $\mathbb{C}^\times$).

(a) Show that the formula

$$f\partial_t \circ gt^\alpha(dt)^\beta = (fg' + \alpha t^{-1}fg + \beta f'g)t^\alpha(dt)^\beta$$

defines an action of W on $V_{\alpha,\beta}$.

(b) Choose a basis $v_k := t^{k+\alpha}(dt)^\beta$ of $V_{\alpha,\beta}$. Verify that $L_n(v_k) = -(k + \alpha + (n+1)\beta)v_{k+n}$.

10. Consider the representations $V_{\alpha,\beta}$ of the Witt algebra W defined above.

(a) Find the necessary and sufficient conditions on $(\alpha, \beta, \alpha', \beta')$ under which $V_{\alpha,\beta} \simeq V_{\alpha',\beta'}$.

(b) Find the necessary and sufficient conditions on (α, β) under which $V_{\alpha,\beta}$ is irreducible.