HOMEWORK 4 (DUE MARCH 27)

1. Present and prove the Plücker relations explicitly, following our discussions in Lecture 13.

2. Verify that $\tau = \tau(x_1, x_2, x_3, \ldots)$ satisfies the first nontrivial equation of the KP hierarchy

$$\Big(\left(\partial_{z_1}^4 + 3\partial_{z_2}^2 - 4\partial_{z_1}\partial_{z_3}\right) \tau(x-z)\tau(x+z) \Big)_{|z=0} = 0$$

if and only if the function

$$u := 2\partial_x^2 \log \tau(x, y, t, c_4, c_5, \ldots)$$

satisfies the KP equation

$$\frac{3}{4}\partial_y^2 u = \partial_x \left(\partial_t u - \frac{3}{2} u \cdot \partial_x u - \frac{1}{4}\partial_x^3 u \right).$$

3. Show that the action $\widehat{\mathfrak{gl}}_n \curvearrowright F^{(m)}$ can be uniquely extended to $\widetilde{\mathfrak{gl}}_n \curvearrowright F^{(m)}$ with $d(\psi_m) = 0$.

4. Establish an isomorphism $\widehat{\mathfrak{gl}}_n \simeq (\widehat{\mathfrak{sl}}_n \oplus \mathcal{A})/(K_1 - K_2)$, where $K_1 = (K, 0), K_2 = (0, K)$.

5. (a) Prove that $\mathcal{F} = \Lambda^{\frac{\infty}{2}} V$ is an irreducible representation of the Clifford algebra generated by $\{\hat{v}_j, \check{v}_j\}_{j\in\mathbb{Z}}$.

(b) Compute $\mathrm{Tr}_{\mathcal{F}}(q^{\mathbf{d}} z^{\mathbf{m}})$, where $\mathbf{m}$ is the operator multiplying elements of $\mathcal{F}^{(m)}$ by the number m, while $\mathbf{d}$ is the operator multiplying homogeneous elements by their degree, defined via:

$$\deg(\psi_0) = 0, \quad \deg(\hat{v}_j) = j, \quad \deg(\check{v}_j) = -j\,.$$

6. (a) Using the boson-fermion correspondence $\mathcal{F} \simeq \mathcal{B}$, compute the answer to Problem 5(b) in the bosonic realization.

(b) Deduce the *Jacobi triple product* identity:

$$\prod_{n\geq 0} (1 - q^n z)(1 - q^{n+1} z^{-1})(1 - q^{n+1}) = \sum_{m\in\mathbb{Z}} (-z)^m q^{\frac{m(m-1)}{2}}.$$

(c) Substitute $q = z^3$ to obtain the *Euler's pentagonal* identity:

$$\prod_{n\geq 1} (1 - z^n) = 1 + \sum_{k\geq 1} (-1)^k \left(z^{\frac{k(3k+1)}{2}} + z^{\frac{k(3k-1)}{2}} \right).$$

$7^\star$. This problem (finally) outlines a proof of Theorem 1 from Lecture 5 stating that for a non-degenerate $\mathbb{Z}$-graded Lie algebra $\mathfrak{g}$ and any $n \geq 0$, the restriction

$$(\cdot,\cdot)_\lambda \colon M^+_\lambda[-n] \times M^-_{-\lambda}[n] \longrightarrow \mathbb{C} \text{ is nondegenerate for generic } \lambda \in \mathfrak{h}^*$$

Identifying $M^{\pm}_{\pm\lambda}[\mp n] \simeq U(\mathfrak{n}_\mp)[\mp n]$ and choosing some fixed bases of the latter, this reduces to a non-vanishing of the corresponding determinant, denoted $\det(\cdot,\cdot)^{\mathfrak{g}}_{\lambda,n}$. The key idea will be to degenerate $\mathfrak{g}$ to a "generalized Heisenberg algebra" where the proof is more feasible.

Step 1 (degeneration process): Consider the $\mathbb{Z}$-graded Lie algebra $\mathfrak{g}^\epsilon = \bigoplus_{n\in\mathbb{Z}} \mathfrak{g}^\epsilon_n$ with $\mathfrak{g}^\epsilon_n = \mathfrak{g}_n$ as vector spaces, and with the Lie bracket defined via

$$[x,y]_\epsilon = [x,y]\cdot \epsilon^{1+\delta_{n,0}+\delta_{m,0}-\delta_{n+m,0}} \quad \text{for any} \quad x\in\mathfrak{g}^\epsilon_n, y\in\mathfrak{g}^\epsilon_m$$

For $\epsilon\neq 0$, show that the following linear map is a Lie algebra isomorphism

$$\varphi_\epsilon\colon \mathfrak{g}^\epsilon \to \mathfrak{g} \qquad \text{with} \quad x\mapsto \epsilon^{1+\delta_{n,0}}x \quad \text{for} \quad x\in\mathfrak{g}^\epsilon_n$$

Show that

$$(xv^{+,\mathfrak{g}^\epsilon}_\lambda, yv^{-,\mathfrak{g}^\epsilon}_{-\lambda})_\lambda = (\varphi_\epsilon(x)v^{+,\mathfrak{g}}_{\lambda/\epsilon^2}, \varphi_\epsilon(y)v^{-,\mathfrak{g}}_{-\lambda/\epsilon^2})_{\lambda/\epsilon^2}$$

for any $x\in U(\mathfrak{n}_-), y\in U(\mathfrak{n}_+)$. Restricting to degree $\pm n$ components, deduce:

$$\det(\cdot,\cdot)^{\mathfrak{g}^\epsilon}_{\lambda,n} = \epsilon^N \det(\cdot,\cdot)^{\mathfrak{g}}_{\lambda/\epsilon^2,n}$$

for some $N\in\mathbb{Z}_{\geq 0}$.

Conclusion: Deduce that the leading term of $\det(\cdot,\cdot)^{\mathfrak{g}}_{\lambda,n} = \det(\cdot,\cdot)^{\mathfrak{g}^1}_{\lambda,n}$ equals $\det(\cdot,\cdot)^{\mathfrak{g}^0}_{\lambda,n}$.

Therefore, it suffices to prove the non-vanishing of $\det(\cdot,\cdot)^{\mathfrak{g}^0}_{\lambda,n}$ for generic $\lambda\in\mathfrak{h}^*$.

Step 2 (degenerated version explicitly): Note that $\mathfrak{g}^0 = \bigoplus_{n\in\mathbb{Z}}\mathfrak{g}_n$ as vector spaces with

$$[x,y]_{\mathfrak{g}^0} = \begin{cases} [x,y] & \text{if } \deg(x)+\deg(y)=0 \\ 0 & \text{otherwise} \end{cases}$$

for homogeneous elements x,y (hence, we call $\mathfrak{g}^0$ a "generalized Heisenberg algebra"). Note that $\mathfrak{n}_\pm = \bigoplus_{n>0}\mathfrak{g}^0_{\pm n}$ are abelian, so that $U(\mathfrak{n}_\pm)\simeq S(\mathfrak{n}_\pm)$.

Verify that the $\mathfrak{g}$-invariant form $(\cdot,\cdot)^{\mathfrak{g}^0}_\lambda\colon S(\mathfrak{n}_-)\times S(\mathfrak{n}_+)\to\mathbb{C}$ is given by

$$(\star) \qquad (a_1\dots a_k, b_1\dots b_l) = \delta_{k,l}\sum_{\sigma\in S(k)} \lambda([a_1,b_{\sigma(1)}])\cdots\lambda([a_k,b_{\sigma(k)}]) \quad \text{with} \quad \lambda_{|\mathfrak{g}_{\neq 0}} = 0$$

Step 3 (verification for $\mathfrak{g}^0$): Use formula $(\star)$ to show that $\det(\cdot,\cdot)^{\mathfrak{g}^0}_{\lambda,n}\neq 0$ for generic $\lambda\in\mathfrak{h}^*$.

8. (a) Show that the leading h-power of $det_n(c,h)$ arises only from the diagonal.

(b) Prove the formula of Lecture 15 for the coefficient of the leading power of h in $\det_m(c,h)$:

$$K_n = \prod_{r,s\geq 1}^{rs\leq n}\Big((2r)^s\cdot s!\Big)^{m(r,s)}$$

with $m(r,s) = p\,(n-rs) - p\,(n-r(s+1))$, where $p(k)$ is the partition function.