HOMEWORK 7

1. Find the Weyl groups of the classical simple Lie algebras of types A_n, B_n, C_n, D_n.

2. Let $\mathfrak{g}(A) = \widehat{\mathfrak{g}}$ be the affinization of a simple finite dimensional Lie algebra $\mathfrak{g}$. Set $\widetilde{\mathfrak{g}} := \mathbb{C}d \ltimes \widehat{\mathfrak{g}}$ and define the category $\mathcal{O}$ accordingly. For a module $V \in \mathcal{O}$ of level k, verify the following relation between the actions of the Casimir operator Δ and the Sugawara operator L_0 on V:

$$\Delta = 2(k + h^\vee)(L_0 + d).$$

3. Derive the Jacobi triple product identity from the Weyl-Kac denominator formula for $\widetilde{\mathfrak{sl}}_2$.

4. Use the explicit realization of $L_d = L_{\omega_0}$ from [Homework 6, Problem 8] to re-derive

$$\mathrm{ch}_{L_d}(e^h) = \frac{\Theta_{0,1}(\tau, z, u)}{\varphi(q)} \qquad \text{for} \quad h = 2\pi i \left(\tfrac{z}{2}\alpha - \tau d + uK\right),\ q = e^{2\pi i \tau}.$$

5. (a) Prove the following product formula for theta functions $\Theta_{n,m} = \Theta_{n,m}(\tau, z, u)$:

$$\begin{aligned} \Theta_{n,m} \cdot \Theta_{n',m'} &= \sum_{j \in \mathbb{Z} \bmod (m+m')\mathbb{Z}} d_j^{(m,m',n,n')}(q) \Theta_{n+n'+2mj, m+m'}, \\ d_j^{(m,m',n,n')}(q) &:= \sum_{k \in \frac{m'n - mn' + 2jmm'}{2mm'(m+m')} + \mathbb{Z}} q^{mm'(m+m')k^2}. \end{aligned}$$

(b) Let $\lambda = md + \frac{n}{2}\alpha \in P_+$ with integers $m \geq n \geq 0$. Use part (a) to prove:

$$\begin{aligned} &\mathrm{ch}_{L_d}(e^h)\mathrm{ch}_{L_\lambda}(e^h) = \sum_{k \in I} \psi_{m,n,k}(q) \mathrm{ch}_{L_{d+\lambda-k\alpha}}(e^h), \\ &I := \left\{ k \in \mathbb{Z} \,\middle|\, -\frac{m-n+1}{2} \leq k \leq \frac{n}{2} \right\}, \\ &\psi_{m,n,k}(q) := \frac{f_k^{(m,n)}(q) - f_{n+1-k}^{(m,n)}(q)}{\varphi(q)}, \\ &f_k^{(m,n)}(q) := \sum_{j \in \mathbb{Z}} q^{(m+2)(m+3)j^2 + ((n+1)+2k(m+2))j + k^2}. \end{aligned}$$

(c) For m, n, k as in part (b), define $r := n+1, s := n+1-2k$ for $k \geq 0$ and $r := m-n+1, s := m-n+2+2k$ for $k < 0$. Prove:

$$\begin{aligned} &\varphi(q) \cdot q^{-k^2} \cdot \psi_{m,n,k}(q) = A + B + C, \text{ where} \\ &A := 1 - q^{rs} - q^{(m+2-r)(m+3-s)}, \\ &B := \sum_{j>0} q^{(m+2)(m+3)j^2 + ((m+3)r - (m+2)s)j} \left(1 - q^{2(m+2)sj + rs}\right), \\ &C := \sum_{j>0} q^{(m+2)(m+3)j^2 - ((m+3)r - (m+2)s)j} \left(1 - q^{2(m+2)(m+3-s)j + (m+2-r)(m+3-s)}\right). \end{aligned}$$

6. Complete the proof of the main Theorem from Lecture 28 by proving the following:

(a) Show that the leading term of det $\big(\langle\cdot,\cdot\rangle^{\eta}\big)$ equals $\prod_{\alpha\in\Delta^+}^{n\geq 1} h_\alpha^{P(\eta-n\alpha)}$, up to a nonzero constant factor.

(b) Let V be a finite dimensional space and $\{H_s\}_{s\in S}$ be a countable union of hyperplanes in V defined by linear functions $f_s\in\mathbb{C}[V]$. Let $F\in\mathbb{C}[V]$ be such that the zero set $Z(F)\subset V$ is contained in the union $\bigcup_{s\in S} H_s$. Show that F is a product of some linear functions f_s (possibly with multiplicities), up to a nonzero constant factor.

(c) For $\alpha\in\Delta^+$ with $(\alpha,\alpha)\neq 0$, establish the linear independence of the functions $\{\phi_\beta(\cdot)\}_{\beta\in\mathbb{Q}\alpha\cap Q^+}$ defined via $\phi_\beta(\eta):=P(\eta-\beta)$ for $\eta\in Q^+$.

7. (a) Prove the following inequality for any simple finite dimensional Lie algebra $\mathfrak{g}=\mathfrak{g}(A)$:

$$\dim\mathrm{Hom}_{\mathfrak{g}}(M_\mu,M_\lambda)\leq 1\qquad \forall\,\lambda,\mu\in\mathfrak{h}^*.$$

(b) Show that the above inequality does not always hold for a general Kac-Moody $\mathfrak{g}(A)$.