

- Last time: • $F \rightarrow \mathfrak{g}^*$ got upgraded to $P \rightarrow \mathfrak{g}_{\text{ext}}^*$
- introduced category $\mathcal{O}$ using $\mathfrak{g}_{\text{ext}}(A)$
- (our exposition slightly differs from [Feigin-Zelevinsky] in that they do not extend $\mathfrak{g}(A)$ but have extra data of P -grading)

• Today: non-degenerate pairings.

Goal: Given $A \in \text{Mat}_{\text{fin}}(\mathbb{C})$ decide if both $\mathfrak{g}(A)$ and $\bar{\mathfrak{g}}(A)$ (defined solely by generators and relations, see Lecture 20) can be equipped with symmetric invariant pairings of degree zero

$$(x, y) = 0 \text{ if } \deg(y) \neq -\deg(x)$$

Minor Assumption: The pairings $d_i := (e_i, f_i) \neq 0 \forall i$

Lemma 1: For such a pairing to exist, matrix A must be symmetrizable.

$$\left. \begin{array}{l} (h_i, h_j) = (h_i, [e_j, f_j]) \xrightarrow{\text{invariance}} ([h_i, e_j], f_j) = a_{ij} \cdot d_j \\ \text{if symmetric} \\ (h_j, h_i) \xrightarrow{\text{is above}} a_{ji} \cdot d_i \end{array} \right\} \Rightarrow A \cdot \begin{pmatrix} d_1 & 0 \\ 0 & d_n \end{pmatrix} \text{ -symmetric} \\ \Downarrow \\ \begin{pmatrix} d'_1 & 0 \\ 0 & d'_n \end{pmatrix} \cdot A \text{ -symmetric}$$

Exercise: Show that if A is indecomposable, then diagonal matrix D s.t. $A \cdot D$ -symmetric is unique up to scaling.

Lemma 2: If A is indecomposable symmetrizable, then there is at most one (up to scaling) symmetric invariant form on $\bar{\mathfrak{g}}(A), \mathfrak{g}(A)$ of degree 0

Let $\mathfrak{g}$ be either $\bar{\mathfrak{g}}(A), \mathfrak{g}(A)$. A bilinear pairing $(,): \mathfrak{g} \times \mathfrak{g} \rightarrow \mathbb{C}$ naturally gives rise to $\gamma: \mathfrak{g} \rightarrow \mathfrak{g}^*$ (=restricted dual, where we note that $\dim \mathfrak{g}_\alpha < \infty \forall \alpha \in \mathbb{Q}$)

$$x \mapsto (y \mapsto (x, y))$$

Moreover:

$(,)$ -invariant $\Rightarrow \gamma$ is a $\mathfrak{g}$ -module morphism

(indeed: $\gamma(\text{ad}(z)x)(y) = ([z, x], y) = -(x, [z, y]) = \text{ad}^*(z)(\gamma(x))(y)$)

But $\mathfrak{g}$ is generated by $\{e_i, f_i\}$, hence, γ is uniquely determined by $\gamma(e_i), \gamma(f_i)$. However, being of degree zero and symmetric, we get

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$$\gamma(e_i) = d_i f_i^*, \quad \gamma(f_i) = d_i e_i^* \quad \text{for some } \{d_i\}$$

However, evoking the argument from the proof of Lemma 1, we see that $A \cdot D$ must be symmetric, hence, D is unique up to scaling by Exercise

Lemma 3: If A is indecomposable, symmetrizable, and $(,) : \mathfrak{g}(A) \times \mathfrak{g}(A) \rightarrow \mathbb{C}$ is a symmetric invariant degree 0 pairing, then its kernel $\text{Ker} = \{x \in \mathfrak{g}(A) \mid (x, y) = 0 \ \forall y\}$ coincides with the center of $\mathfrak{g}(A)$.

Note that Ker of invariant pairing is an ideal (as $([z, x], y) = (x, -[z, y]) = 0$). Also being of degree 0, it's clear that Ker is a graded ideal. But due to Lemma 3(c) of Lecture 22, we then must have $\text{Ker} \subseteq \mathbb{Z} = \text{center of } \mathfrak{g}(A)$.

However, by Lemma 3(a) of Lecture 22: $\mathbb{Z} = \{\sum \beta_i h_i \mid \sum \beta_i a_{ij} = 0 \ \forall j\}$. As $(,)$ is of degree zero, it suffices to show $(\sum \beta_i h_i, h_j) = 0 \ \forall j$, which follows from $(\sum \beta_i h_i, h_j) = (\sum \beta_i a_{ij}) d_j = 0$.

The main result for today is:

Theorem 1: If A -indecomposable symmetrizable, then there exists a nonzero symmetric invariant pairing of degree 0 on $\mathfrak{g}(A)$ and $\mathfrak{g}(A)$.

Note that it suffices to prove existence of such form on $\mathfrak{g}(A)$, as then we can just take the pull-back along the projection $\mathfrak{g}(A) \times \mathfrak{g}(A) \rightarrow \mathfrak{g}(A) \times \mathfrak{g}(A)$. The above result is Proposition 4.1 of [Feigin - Zelerovsky] and we shall closely follow its proof from §47.

Let $\mathfrak{g}^k := \bigoplus_{\substack{\alpha \in \Delta \cup \{0\} \\ |\alpha| \leq k}} \mathfrak{g}_\alpha$, where Δ denotes the set of roots of $\mathfrak{g} = \mathfrak{g}(A)$ and the height $|c_1 \alpha_1 + \dots + c_n \alpha_n| := c_1 + \dots + c_n$.

Note that $\mathfrak{g}^1 \subseteq \mathfrak{g}^2 \subseteq \mathfrak{g}^3 \subseteq \dots$ and $\mathfrak{g} = \bigcup \mathfrak{g}^k$. We shall construct pairing $(,) : \mathfrak{g}^k \times \mathfrak{g}^k \rightarrow \mathbb{C}$ inductively.

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We shall inductively construct pairings $(,): \mathfrak{g}^k \times \mathfrak{g}^k \rightarrow \mathbb{C}$ which are symmetric of degree 0, and invariant in the sense $([x, y], z) = (x, [y, z]) \forall x, y, z \in \mathfrak{g}^k$ s.t. $[x, y], [y, z] \in \mathfrak{g}^k$, and restriction to $\mathfrak{g}^{k-1} \times \mathfrak{g}^{k-1}$ recovers previous step.

Base of Induction: $k=1$

Note that $\mathfrak{g}^1 = \mathfrak{h} \oplus \bigoplus_{i=1}^n \mathbb{C} e_i \oplus \bigoplus_{i=1}^n \mathbb{C} f_i$. Following above discussions, pick $\{d_i\}$ s.t. $A = \begin{pmatrix} d_1 & & 0 \\ & \ddots & \\ 0 & & d_n \end{pmatrix}$ -symmetric, and degree $(,)$ on $\mathfrak{g}^1 \times \mathfrak{g}^1$ via:

$$(e_i, f_i) = d_i, (h_i, h_j) = a_{ij} d_j = a_{ji} d_i \text{ and all other pairings vanish:} \\ (e_i, e_j) = 0 = (f_i, f_j), (h_i, e_j) = 0 = (h_i, f_j), (e_i, f_{j \neq i}) = 0$$

It's clearly of degree 0. To verify invariance, we need two checks:

$$([h_i, e_j], f_j) = a_{ij} \cdot (e_j, f_j) = a_{ij} d_j = (h_i, h_j) = (h_i, [e_j, f_j])$$

$$([e_j, h_i], f_j) = -a_{ij} (e_j, f_j) = (e_j, [h_i, f_j])$$

$$([h_i, f_j], e_j) = -a_{ij} d_j = -(h_i, h_j) = (h_i, [f_j, e_j])$$

Step of Induction

Assume we already have the desired $(,): \mathfrak{g}^k \times \mathfrak{g}^k \rightarrow \mathbb{C}$, and we want to extend to $\mathfrak{g}^{k+1} \times \mathfrak{g}^{k+1} \rightarrow \mathbb{C}$. Let $\alpha \in \Delta_+$ be such that $|\alpha| = k+1$ and pick any $x \in \mathfrak{g}_\alpha, y \in \mathfrak{g}_{-\alpha}$. For degree reasons, it suffices to define $(x, y) = (y, x)$ in the above setup. To this end, as $\bigoplus_{\alpha \in \Delta_+} \mathfrak{g}_\alpha$ is generated by $\{e_i\}$, we pick a presentation $x = \sum_s [a_s, b_s]$ (which is non-unique!) with $a_s \in \bigoplus \mathbb{C} e_i \subset \mathfrak{g}^1$ and $b_s \in \bigoplus_{|\beta|=k} \mathfrak{g}_\beta \subseteq \mathfrak{g}^k$. Then, note that $[b_s, y] \in \bigoplus \mathbb{C} f_i$ for degree reasons.

$$\text{Set: } (x, y) = (y, x) := \sum_s (a_s, [b_s, y])$$

• First, we need to show this is independent of presentation $x = \sum [a_s, b_s]$. To this end, it suffices to verify that if $0 = \sum [a_s, b_s]$ with a_s, b_s as above, then $\sum (a_s, [b_s, y]) = 0$, which will be derived from Jacobi + induct hypothesis.

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We can likewise present y as $y = \sum_t [u_t, v_t]$ with $u_t, v_t \in \mathfrak{g}^k$; and it suffices to treat a single summand: $y = [u, v]$.

$$\begin{aligned} \text{So: } \sum_s (a_s, [b_s, y]) &= \sum_s (a_s, [b_s, [u, v]]) \stackrel{\text{Jacobi}}{=} \sum_s (a_s, [b_s, u], v) + (a_s, [u, [b_s, v]]) \\ &\stackrel{\text{induction hypothesis}}{=} \sum_s ([a_s, [b_s, u]], v) + ([b_s, v], [a_s, u]) \stackrel{\text{induction hypothesis}}{=} \\ &= \sum_s (v, [a_s, [b_s, u]]) + (v, [a_s, u], b_s) \stackrel{\text{Jacobi}}{=} (v, [\sum_s [a_s, b_s], u]) = 0 \end{aligned}$$

Warning: Every time we use induction we need to check arguments $\in \mathfrak{g}^k$. ✓

• Finally, it remains to verify that the defined extension of $(\cdot, \cdot)$ from $\mathfrak{g}^k \times \mathfrak{g}^k$ to $\mathfrak{g}^{k+1} \times \mathfrak{g}^{k+1}$ is invariant. This relies on:

$$(1) ([x, y], z) = (x, [y, z]) \quad \forall x \in \mathfrak{g}_{\alpha-\beta}, y \in \mathfrak{g}_{\beta}, z \in \mathfrak{g}_{-\alpha} \text{ with } |\alpha| = k+1, |\beta| \leq k, \alpha, \beta \in \Delta_+$$

which essentially follows from our definition (where we could just assume $a_s, b_s \in \mathfrak{g}^k$ rather than $a_s \in \mathfrak{g}^1, b_s \in \mathfrak{g}^k$)

$$(2) ([x, y], z) = (x, [y, z]) \quad \forall x \in \mathfrak{g}_{\alpha}, y \in \mathfrak{g}_{-\beta}, z \in \mathfrak{g}_{-\alpha+\beta} \text{ with } |\alpha| = k+1, |\beta| \leq k+1, \alpha, \beta \in \Delta_+$$

To this end, it suffices to treat $x = [a, b]$ with $a, b \in \mathfrak{g}^k$ (as the general x is the sum of such). Then, as a check of well-definedness:

$$\begin{aligned} ([x, y], z) &\stackrel{\text{Jacobi}}{=} ([a, [b, y]], z) + ([a, y], [b, z]) \stackrel{\text{induction hypothesis}}{=} (a, [b, [y, z]]) + [b, [z, [a, y]]) \\ &\stackrel{\text{induction}}{=} (a, [b, [y, z]]) + (a, [y, [b, z]]) \stackrel{\text{Jacobi}}{=} (a, [b, [y, z]]) \stackrel{\text{def'n}}{=} (x, [y, z]) \end{aligned}$$

This completes the step of induction. As $k \rightarrow \infty$, we get the desired symmetric invariant free $\mathcal{O}$ pairing on $\mathfrak{g}$. ✓

Recall now $F := \mathbb{Q} \otimes_{\mathbb{Z}} \mathbb{C}$ with basis addition and define a linear map

$$\gamma: F \rightarrow \mathfrak{h} \text{ via } \alpha_i \mapsto d_i^{-1} h_i =: h_{\alpha_i}$$

Def 1: For any $\alpha \in F$, set $h_{\alpha} := \gamma(\alpha)$

Lemma 4: $(h_{\alpha}, h) = \bar{\alpha}(h) \quad \forall \alpha \in F, h \in \mathfrak{h}$

► Suffices to check on basis:

$$(h_{\alpha_i}, h_j) = d_i^{-1}(h_i, h_j) = a_{ji} = \bar{\alpha}_i(h_j) \quad \square$$

Lemma 5: $\forall \alpha \in \Delta_+$, $\forall x \in \mathfrak{g}_{\alpha}, y \in \mathfrak{g}_{-\alpha}$ have: $[x, y] = (x, y) h_{\alpha}$

► As $(,)$ was constructed by induction on $|\alpha|$, we shall likewise prove Lemma

Base: $|\alpha| = 1 \Rightarrow \alpha = \alpha_i$, but then $[e_i, f_i] = h_i = d_i \cdot d_i^{-1} h_i = d_i h_{\alpha_i}$

Step of Induction: Suffices to assume $x = [a, b]$, with $a \in \mathfrak{g}_{\beta}, b \in \mathfrak{g}_{\gamma}$, where

$$\begin{aligned} [x, y] &= [[a, b], y] \stackrel{\text{Jacobi}}{=} [[a, y], b] + [a, [b, y]] \stackrel{\text{induction}}{=} -(b, [a, y]) h_{\gamma} \\ &\quad + (a, [b, y]) h_{\beta} = \\ &\stackrel{\text{invariance}}{=} ([a, b], y) h_{\beta} + ([a, b], y) h_{\gamma} = ([a, b], y) h_{\alpha} = (x, y) h_{\alpha} \quad \square \end{aligned}$$

If we endow $\mathfrak{g}(A)$ with the principal $\mathbb{Z}$ -grading via $\deg(e_i) = 1 = -\deg(f_i) \forall i$ then $\mathfrak{g}(A)$ becomes a $\mathbb{Z}$ -graded Lie algebra with $\mathfrak{g}(A)[0] = \mathfrak{h}$
 $\mathfrak{g}(A)[\pm n] = \bigoplus_{\alpha \in \Delta_+} \mathfrak{g}(A)[\pm \alpha]$

Corollary 1: $\mathfrak{g}(A)$ is a nondegenerate $\mathbb{Z}$ -graded Lie algebra, see Lecture 3

► By Lemma 5, if $\xi \in \mathfrak{h}^*$ is such that $\xi(h_{\alpha}) \neq 0 \quad \forall \alpha \in \Delta_+$, then it's clear that

$$\begin{aligned} \mathfrak{g}(A)[n] \times \mathfrak{g}(A)[-n] &\rightarrow \mathbb{C} \text{ is non-degenerate} \\ (x, y) &\longmapsto \xi(x, y) \end{aligned}$$

Thus, $\mathfrak{g}(A)$ fits into the class of Lie algebras we discussed in the very beginning of the course. However, the pairing $(,): \mathfrak{g}(A) \times \mathfrak{g}(A) \rightarrow \mathbb{C}$ is degenerate (with kernel $\text{Ker} = \mathbb{Z}$, see Lemma 3). We shall now fix this by using $\mathfrak{g}_{\text{ext}}(A)$ instead.

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Recall $P = \mathfrak{h}^* \oplus F$, which we identified with $\mathfrak{h}_{\text{ext}}^*$ in Lecture 22.

Define a symmetric pairing

$$(\cdot, \cdot): P \times P \rightarrow \mathbb{C} \quad \text{by} \quad (\varphi + \alpha, \psi + \beta) = \varphi(h_\beta) + \psi(h_\alpha) + (h_\alpha, h_\beta) \\ \forall \varphi, \psi \in \mathfrak{h}^*, \forall \alpha, \beta \in F$$

Note: $(\mathfrak{h}^*, \mathfrak{h}^*) = 0$, but $(\cdot, \cdot)$ is nondegenerate pairing on P .

Exercise: Identifying $P \cong \mathfrak{h}_{\text{ext}}^*$ and further using this pairing to identify $\mathfrak{h}_{\text{ext}}^* \cong \mathfrak{h}_{\text{ext}}$, verify that the resulting pairing on $\mathfrak{h}_{\text{ext}}$ is given by:

$$(D_i, D_j) = 0, \quad (D_i, h_{\alpha_j}) = \delta_{ij}, \quad (h_{\alpha_i}, h_{\alpha_j}) = d_i^{-1} a_{ij}.$$

Corollary 2: The resulting extension of $(\cdot, \cdot)$ on $\mathfrak{g}_{\text{ext}}(A)$ is a nondegenerate symmetric invariant pairing