Lecture #26

Last time: Weyl-Kac character formula for $ch(L_\lambda)$, where $\lambda \in P_+ = \{\text{dominant integral weights}\}$

Recall that W was a subgroup of $GL(P)$, where $P = \mathfrak{h}^* \oplus F$ with $F = \mathbb{C}\alpha_1 \oplus \ldots \oplus \mathbb{C}\alpha_n$ generated by simple reflections $\boxed{r_i: \chi \mapsto \chi - \chi(h_i)\alpha_i \quad \forall \chi \in P}$ Even though it's defined as a subgroup of linear maps $P \to P$ (i.e. can be represented by $2n\times 2n$ matrices), it can be often realized on a smaller dimensional vector space. For example, as we shall now see W can be realized inside $\text{End}(F)$ if $\mathfrak{g}$-simple f.d.

Recall: • $F \subseteq P$ is W-invariant, and explicitly $r_i(\alpha_j) = \alpha_j - a_{ij}\alpha_i$
• the resulting action $W \curvearrowright P/F \simeq \mathfrak{h}^*$ is trivial.

Lemma 1: If $\mathfrak{g}$ is a simple fin. dim. Lie algebra, then the restriction $w \mapsto w_{|F}$ establishes isom. btw $W \subseteq \text{End}(P)$ and the group generated by $r_{i|F}$ in $\text{End}(F)$.

Thus, for simple Lie algebras, there is no need to treat P instead of $F = Q \underset{\mathbb{Z}}{\otimes} \mathbb{C}$.
For any $\chi \in P$, consider a function $f_\chi: W \to F$, $w \mapsto \chi - w\chi$. This function clearly satisfies $f_\chi(g_1g_2) = f_\chi(g_1) + g_1 f_\chi(g_2)$ $\forall g_1, g_2 \in W$. In particular, given $\chi_1, \chi_2 \in P$ we note that $f_{\chi_1} = f_{\chi_2} \iff f_{\chi_1}(r_i) = f_{\chi_2}(r_i)$ $\forall\, 1 \le i \le n \iff \bar\chi_1 = \bar\chi_2 \in \mathfrak{h}^*$

Proof of Lemma 1

► We just need to verify that if $w \in W$ satisfies $w_{|F} = id_F$, then $w = id_P$.
But by Lemma 2 of Lecture 22, if $\mathfrak{g} = \mathfrak{g}(C)$-simple f.d. (i.e. C-nondeg. positive) then $\forall \chi_1 \in P$ $\exists \chi_2 \in F$ s.t. $\bar\chi_1 = \bar\chi_2$. But then by the sentence preceding the proof: $f_{\chi_1} = f_{\chi_2} \Rightarrow \chi_1 - w\chi_1 = \chi_2 - w(\chi_2) \underset{\text{as } w_{|F} = id_F}{=} 0 \Rightarrow \chi_1 = w\chi_1$ $\forall \chi \Rightarrow w = id_P$ ∎

Corollary 1: If $\mathfrak{g}$-simple f.dim Lie algebra, then Weyl gp W is finite.

► As the set of roots Δ of roots is finite, spans F, and is W-invariant by Lemma 2 of Lecture 25, W is a subgp of permutations of $\Delta \Rightarrow |W| < \infty$ ∎

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[Exercise: Compute Weyl groups of classical simple Lie algebras (A_n, B_n, C_n, D_n)

By the same reasoning as used in the proof of Lemma 1, we see that for untwisted affine Kac-Moody algebras $\mathfrak{g}(\hat{A})$ that could be realized as $\mathfrak{g}[t,t^{-1}]\oplus\mathbb{C}K=\tilde{\mathfrak{g}}$, see Theorem 1 of Lecture 21, it suffices to treat $\hat{\mathfrak{g}}=\mathbb{C}d\ltimes\tilde{\mathfrak{g}}$ instead of adding many more degree elements as in $\mathfrak{g}^{ext}(\hat{A})$ from Lecture 22. In particular, $\hat{W}$ can be realized as a subgp of $\mathrm{End}(\mathfrak{h}\oplus\mathbb{C}K\oplus\mathbb{C}d)$, where $\mathfrak{h}\subseteq\mathfrak{g}$ - Cartan. For this week, we shall primarily need $\hat{\mathfrak{sl}}_2$ and $\tilde{\mathfrak{sl}}_2$. So let's see what $\hat{W}$ looks like in this case.

Recall: $\alpha_0=K-\alpha_1$, $(\alpha_1,\alpha_1)=2$, $(d,K)=(K,d)=1$, all other pairings $=0$.
Let's compute how simple reflections $s_0=s_{\alpha_0}$, $s_1=s_{\alpha_1}$ look like
Let $\alpha=\alpha_1$.

s_1: $\alpha\mapsto\alpha-(\alpha,\alpha)\alpha=-\alpha$, $K\mapsto K-(\alpha,K)\alpha=K$, $d\mapsto d-(\alpha,d)\alpha=d$

s_0: $\alpha\mapsto\alpha-(K-\alpha,\alpha)(K-\alpha)=\alpha+2(K-\alpha)=-\alpha+2K$, $K\mapsto K$,
$d\mapsto d-(K-\alpha,d)(K-\alpha)=d-K+\alpha$

so that in the basis $\{\alpha,K,d\}$ we get

$$\begin{cases} s_1:\ \alpha\mapsto-\alpha,\ K\mapsto K,\ d\mapsto d \\ s_0:\ \alpha\mapsto-\alpha+2K,\ K\mapsto K,\ d\mapsto d-K+\alpha \end{cases}$$

Also we know (and easily checked from above f-las) that $s_0^2=s_1^2=id$.

Let's compute $t_1:=s_1s_0$: $\alpha\mapsto\alpha+2K$, $K\mapsto K$, $d\mapsto d-\alpha-K$

$t_{-1}:=t_1^{-1}=s_0s_1$: $\alpha\mapsto\alpha-2K$, $K\mapsto K$, $d\mapsto d+\alpha-K$

As $s_0^2=s_1^2=id$, any element in $\hat{W}$ can be written either as

$\underbrace{\underbrace{s_1s_0}_{t_1}\underbrace{s_1s_0}_{t_1}\ldots\underbrace{s_1s_0}_{t_1}}_{0\le k\text{ times}}=t_1^k=:t_k$, $\underbrace{\underbrace{s_0s_1}_{t_{-1}}\underbrace{s_0s_1}_{t_{-1}}\ldots\underbrace{s_0s_1}_{t_{-1}}}_{k>0\text{ times}}=t_{-1}^k=t_{-k}$,

or $\underbrace{s_0s_1s_0\ldots s_1s_0}_{2k+1,\ k\ge0}=s_1t_{k+1}$, or $\underbrace{s_1s_0s_1\ldots s_0s_1}_{2k+1,\ k\ge0}=s_1t_{-k}$.

UPSHOT: $\hat{W}=\{t_k,\ s_1t_k\mid k\in\mathbb{Z}\}$ with $\det(s_1)=-1$, $\det(t_k)=1$.

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Explicitly, evoking the formula for t_k-action, we obtain:

$$\boxed{\begin{array}{l} t_k: \alpha \mapsto \alpha + 2k\cdot K,\ K \mapsto K,\ d \mapsto d - k\cdot\alpha - k^2\cdot K \\ s_1 t_k: \alpha \mapsto -\alpha + 2k\cdot K,\ K \mapsto K,\ d \mapsto d + k\alpha - k^2 K \end{array} \quad \forall k \in \mathbb{Z}}$$

Remark: In general, the Weyl group of $\hat{\mathfrak{g}}$ will be the semidirect product $W^{fin} \ltimes Q^\vee$, where W^{fin} is the Weyl gp of underlying simple $\mathfrak{g}$, and $Q^\vee \simeq \mathbb{Z}^{\dim(\mathfrak{h})}$ - coroot lattice of $\mathfrak{g}$. In the case of $\hat{\mathfrak{sl}}_2$, this is $\mathbb{Z}/2\mathbb{Z} \ltimes \mathbb{Z}$ with $(0,k) \leftrightarrow t_k$, $(1,k) \leftrightarrow s_1 t_k$.

The main goals for this week are to finally establish two earlier results:

1) Unitarity of Vir-modules $L_{h,c}$ at the discrete series, i.e.

$$\boxed{c = c(m) = 1 - \frac{6}{(m+2)(m+3)}, \quad h = h_{r,s}(m) = \frac{((m+3)r - (m+2)s)^2 - 1}{4(m+2)(m+3)}}$$

where $r, s, m \in \mathbb{Z}_{\geq 0}$ satisfying $1 \leq s \leq r \leq m+1$, cf. Theorem 3 of Lect. 8.

2) Kac's lemma on zeroes of $\det_m(c,h)$ from Prop 1 of Lecture 15:

$$\boxed{\begin{array}{l} \det_{rs}(c, h_{r,s}(c)) = 0 \quad \forall r,s \in \mathbb{Z}_{\geq 1}, \text{ where} \\ h_{r,s}(c) = \frac{1}{48}\left((13-c)(r^2+s^2) + \sqrt{(c-1)(c-25)}\,(r^2-s^2) - 24rs - 2 + 2c\right) \end{array}}$$

To this end, we shall utilize the Goddard - Kent - Olive construction of Lecture 19 with $\mathfrak{a} = \mathfrak{sl}_2$ and modules V', V'' - irreducible unitary highest weight modules of $\hat{\mathfrak{sl}}_2$ (or $\tilde{\mathfrak{sl}}_2$). Note: the unitarity allows to decompose into the irreducibles, and we shall calculate their multiplicities using the Weyl-Kac formula of Lecture 25 together with explicit description of the Weyl gp from p.2.

Recall the Weyl-Kac character formula, written as follows:

$$\boxed{ch_{L_\lambda} = \frac{\sum_{w\in W} \det(w)\, e^{w(\lambda+\rho)}}{\sum_{w\in W} \det(w)\, e^{w(\rho)}}}$$

Note that while ch_V is a formal expression it basically allows to compute $Tr_V(e^h) = \sum_\mu \dim(V[\mu])\cdot e^{\mu(h)}$ $\forall\, h \in \mathfrak{h}^*_{ext}$. In the present context, we shall be working with $\hat{sl}_2$ instead of $\mathfrak{g}_{ext}$, as noted on page 2.

Recall: • $W = \{t_k,\ s_1 t_k \mid k\in\mathbb{Z}\}$, $\mathfrak{h}\simeq\mathfrak{h}^*$ has a basis $\{\alpha, K, d\}$

• fundamental weights ω_0, ω_1 given by $\omega_i(h_j)=\delta_{ij}$ are $\begin{cases}\omega_1 = \frac{\alpha}{2}+d\\ \omega_0 = d\end{cases}$

• finally ρ can be chosen as $\rho = \omega_0+\omega_1 = \frac{1}{2}\alpha + 2d$.

Let's evaluate both numerator and denominator of above ratio evaluated on general element $h\in\mathfrak{h}$ which we shall present as

$$\boxed{h = 2\pi i\left(\frac{z}{2}\alpha - \tau\cdot d + u\cdot K\right), \quad z,\tau,u\in\mathbb{C}}$$

Goal: Evaluate $\sum_{w\in W}\det(w)\, e^{(w(\mu),h)}$ for any $\boxed{\mu = md + \frac{n}{2}\alpha + rK}$

• if $w = t_k \Rightarrow \det(w) = 1$ and $w(\mu) = m(d - k\alpha - k^2K) + \frac{n}{2}(\alpha + 2k\cdot K) + rK$
$= \alpha\cdot\left(\frac{n}{2} - mk\right) + d\cdot m + K(r + kn - k^2 m)$

$\Rightarrow (w(\mu), h) = 2\pi i\left(z\left(\frac{n}{2}-mk\right) + mu - \tau(r+kn-k^2m)\right)$

So: $\boxed{\sum_{k\in\mathbb{Z}}\det(t_k)\, e^{(t_k(\mu),h)} = \sum_{k\in\mathbb{Z}} e^{2\pi i\left(z\left(\frac{n}{2}-mk\right)+mu-\tau(r+kn-k^2m)\right)}}$

• if $w = s_1t_k \Rightarrow \det(w) = -1$ and $w(\mu) = \alpha\left(-\frac{n}{2}+mk\right) + d\cdot m + K\cdot(r+kn-k^2m)$

So: $\boxed{\sum_{k\in\mathbb{Z}}\det(s_1t_k)\, e^{(s_1t_k(\mu),h)} = -\sum_{k\in\mathbb{Z}} e^{2\pi i\left(-z\left(\frac{n}{2}-mk\right)+mu-\tau(r+kn-k^2m)\right)}}$

We shall now rewrite these via theta functions:

Def 1: For any $n\in\mathbb{Z}$, $m\in\mathbb{Z}\setminus\{0\}$, the Jacobi-Riemann theta f-n is

$$\boxed{\Theta_{n,m}(\tau,z,u) := e^{2\pi i m\cdot u}\sum_{k\in\frac{n}{2m}+\mathbb{Z}} e^{2\pi i m(k^2\tau + kz)}}$$

$\leftarrow$ absolutely converges $\forall z,u$ & $\mathrm{Im}(\tau)>0$

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- In the case of $w = t_k$ from previous page, replace $\underset{\Uparrow \atop \mathbb{Z}}{k} \rightsquigarrow \frac{n}{2m} - \underset{\Uparrow \atop \frac{n}{2m}+\mathbb{Z}}{k}$ to get:

$$z\left(\tfrac{n}{2} - mk\right) + mu - \tau(r + kn - k^2 m) \rightsquigarrow z\, mk + m\cdot u + \tau\left(\left(k^2 - \tfrac{n}{m}k + \tfrac{n^2}{4m^2}\right)m - \left(\tfrac{n}{2m} - k\right)n - r\right)$$
$$= mkz + mu + mk^2\tau - \tau\left(r + \tfrac{n^2}{4m}\right), \text{ so that}$$

$$\boxed{\sum_{k\in\mathbb{Z}} \det(t_k)\, e^{(t_k(\mu), h)} = \Theta_{n,m}(\tau, z, u)\cdot e^{2\pi i\tau\left(-r - \frac{n^2}{4m}\right)}}$$

- In the case of $w = s_1 t_k$ from previous page, replace $\underset{\Uparrow \atop \mathbb{Z}}{k} \rightsquigarrow \frac{n}{2m} + \underset{\Uparrow \atop \frac{-n}{2m}+\mathbb{Z}}{k}$ to get

$$-z\left(\tfrac{n}{2} - mk\right) + mu - \tau(r + kn - k^2 m) \rightsquigarrow z\cdot mk + m\cdot u + \tau\left(\left(k^2 + \tfrac{n}{m}k + \tfrac{n^2}{4m^2}\right)m - \left(\tfrac{n}{2m} + k\right)n - r\right)$$
$$= mkz + mu + mk^2\tau - \tau\left(r + \tfrac{n^2}{4m}\right), \text{ so that}$$

$$\boxed{\sum_{k\in\mathbb{Z}} \det(s_1 t_k)\, e^{(s_1 t_k(\mu), h)} = -\Theta_{-n,m}(\tau, z, u)\cdot e^{2\pi i\tau\left(-r - \frac{n^2}{4m}\right)}}$$

Combining the above two boxed formulas and using $q := e^{2\pi i \tau}$, we get:

$$\boxed{\sum_{w\in W} \det(w)\, e^{(w\mu, h)} = q^{-r - \frac{n^2}{4m}}\left(\Theta_{n,m}(\tau, z, u) - \Theta_{-n,m}(\tau, z, u)\right)}$$

Evoking $\rho = 2d + \frac{\alpha}{2}$, the Weyl–Kac character formula implies:

Proposition 1: For any $\lambda = md + \frac{n}{2}\alpha + rK \in P_+$ and $h = 2\pi i\left(\frac{z}{2}\alpha - \tau d + uK\right)$:

$$ch_{L_\lambda}(e^h) = \frac{\Theta_{n+1,m+2}(\tau, z, u) - \Theta_{-n-1,m+2}(\tau, z, u)}{\Theta_{1,2}(\tau, z, u) - \Theta_{-1,2}(\tau, z, u)} \cdot q^{-s_\lambda}$$

where $q = e^{2\pi i \tau}$, $s_\lambda = r + \frac{(n+1)^2}{4(m+2)} - \frac{1}{8}$.

<u>Note</u>: $\lambda = md + \frac{n}{2}\alpha + rK \in P_+ \iff n, m-n \in \mathbb{Z}_{\geq 0}$

in which case $\lambda = (m-n)\omega_0 + n\omega_1 + rK$.

We also note that K acts on L_λ via $(\lambda, K) = m\cdot \mathrm{Id} \Rightarrow L_\lambda$ has level m.

<u>Finally</u>: L_λ – unitary iff $n, m-n \in \mathbb{Z}_{\geq 0}$ and $r \in \mathbb{R}$ (see [Lecture 17, Prop 3])