Lecture #28

Today: Jantzen-Kac-Kazhdan-Shapovalov determinant formula

Let $\mathfrak{g}=\mathfrak{g}(A)$ be a Kac-Moody algebra, and consider its Verma module M_λ.

Q: 1) When M_λ is irreducible?

2) If M_λ is not irreducible, can we describe all its irreducible subquotients?

Recall a technical tool that we introduced back in Lecture 6 to address 1).
The setup there was: $\mathfrak{g}=\bigoplus_{n\in\mathbb{Z}}\mathfrak{g}_n$ ($\mathfrak{g}_0$ - abelian) is the $\mathbb{Z}$-graded Lie algebra, $\omega:\mathfrak{g}\to\mathfrak{g}$ - involution s.t. $\omega(\mathfrak{g}_n)=\mathfrak{g}_{-n}$ $\forall n$, $\omega|_{\mathfrak{g}_0}=-\mathrm{id}$. Then, we endowed the Verma module M_λ with a unique symmetric form

$$\boxed{M_\lambda\times M_\lambda\xrightarrow{(\,,\,)_\lambda}\mathbb{C}\quad \text{s.t. } (v_\lambda,v_\lambda)=1,\ (av,w)_\lambda+(v,\omega(a)w)_\lambda=0 \quad \forall a\in\mathfrak{g}\ \forall v,w\in M_\lambda}$$

The key property of this contravariant = Shapovalov form is:

$$\boxed{M_\lambda\text{- irreducible} \Longleftrightarrow (\,,\,)_\lambda\text{ - nondegenerate}}$$

While M_λ is ∞-dim, but the pairing was graded in that $(xv_\lambda, yv_\lambda)=0$ if $x,y\in U(\mathfrak{n}_-)$ of different $\mathbb{Z}$-degrees. Therefore, the question boils down to non-vanishing of the number of determinants.

Recall: We addressed a similar question for Vir back in Lecture 15.

We shall now change the setup a little bit. First, we want to use a finer grading by lattice Q (not $\mathbb{Z}$), and second, we want to view these determinants "universally". To this end, let:

$$\boxed{\sigma:\mathfrak{g}(A)\circlearrowleft \text{- involutive anti-automorphism s.t. } e_i\mapsto f_i,\ f_i\mapsto e_i,\ \sigma|_{\mathfrak{h}}=\mathrm{id}_{\mathfrak{h}}}$$

We also define a projection

$$\boxed{\pi: U(\mathfrak{g})\to U(\mathfrak{h})\simeq S(\mathfrak{h})\simeq \mathbb{C}[\mathfrak{h}^*]}$$

whose kernel is $\mathfrak{n}_-\cdot U(\mathfrak{g})+U(\mathfrak{g})\cdot\mathfrak{n}_+$ (use PBW for $U(\mathfrak{g})=U(\mathfrak{h})\oplus\begin{pmatrix}\mathfrak{n}_-U(\mathfrak{g})\\+U(\mathfrak{g})\mathfrak{n}_+\end{pmatrix}$)

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Def: Define a bilinear form $\langle\,,\,\rangle : U(\mathfrak{g})\times U(\mathfrak{g}) \longrightarrow U(\mathfrak{h})$ via

$$\boxed{\langle x,y\rangle = \pi(\sigma(x)y)}$$

Note that $\sigma(\sigma(x)y)=\sigma(y)x$ and $\pi(\sigma(z))=\pi(z)$ $\forall x,y,z\in U(\mathfrak{g})$.

Lemma 1:
1) $\langle\,,\,\rangle$ - symmetric
2) $\langle x,y\rangle=0$ $\forall x\in U(\mathfrak{g})_{\mu_1}, y\in U(\mathfrak{g})_{\mu_2}$ with $\mu_1\neq\mu_2\in Q$.
3) $\langle x,y\rangle=0$ if $x\in U(\mathfrak{g})\mathfrak{n}_+$ or $y\in U(\mathfrak{g})\mathfrak{n}_+$

▸ Obvious (exercise)

In particular, $\langle\,,\,\rangle$ is uniquely determined by its restrictions

$$\boxed{\langle\,,\,\rangle^\eta : U(\mathfrak{n}_-)_{-\eta}\times U(\mathfrak{n}_-)_{-\eta} \longrightarrow U(\mathfrak{h}) \quad \forall \eta\in Q_+}$$

Let $P(\mu):=\dim U(\mathfrak{n}_-)_{-\mu}$ $\forall\mu\in Q$ ← called Kostant partition function.

As any element of $U(\mathfrak{h})\simeq S(\mathfrak{h})\simeq \mathbb{C}[\mathfrak{h}^*]$ can be evaluated at $\lambda\in\mathfrak{h}^*$:

Def 2: Define the bilinear form $\langle\,,\,\rangle_\lambda : M_\lambda\times M_\lambda\longrightarrow\mathbb{C}$ via

$$\boxed{\langle u_1 v_\lambda, u_2 v_\lambda\rangle_\lambda := \langle u_1,u_2\rangle(\lambda) \quad \forall u_1,u_2\in U(\mathfrak{n}_-)}$$

Note: By the very definition, we also have $\langle u_1v_\lambda, u_2v_\lambda\rangle_\lambda = \langle u_1,u_2\rangle(\lambda)$ $\forall u_1,u_2\in U(\mathfrak{g})$

Lemma 2:
1) $\langle av,w\rangle_\lambda=\langle v,\sigma(a)w\rangle_\lambda$ $\forall v,w\in M_\lambda$, $a\in U(\mathfrak{g})$
2) $\langle M_\lambda[\lambda-\mu_1], M_\lambda[\lambda-\mu_2]\rangle=0$ $\forall\mu_1\neq\mu_2\in Q_+$
3) $\mathrm{Ker}(M_\lambda\twoheadrightarrow L_\lambda)=\mathrm{Ker}(\langle\,,\,\rangle_\lambda)$
4) Find explicit relation b/w $\langle\,,\,\rangle_\lambda$ and $(\,,\,)_\lambda$ on p.1

Exercise: Prove it

Key Theorem: Up to a nonzero factor:

$$\det(\langle\cdot,\cdot\rangle^\eta)=\prod_{\alpha\in\Delta^+}\prod_{n\geq 1}\left(h_\alpha+\rho(h_\alpha)-n\cdot\frac{(\alpha,\alpha)}{2}\right)^{P(\eta-n\alpha)}$$

↑ roots are counted with multiplicities!

Corollary 1: M_λ - irreducible $\iff$ $(\lambda+\rho)(h_\alpha)\neq\frac{n}{2}(\alpha,\alpha)$ $\forall\alpha\in\Delta^+, n\in\mathbb{Z}_{>0}$

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The proof of the theorem involves several "old" ideas that we discuss now as well as another "new" idea that we shall discuss next time.

- First of all, computing the leading term of the polynomial $\det(<,>^\eta)$ is rather straightforward, and is similar to Virasoro case, see Problems 7&8 in Homework 4.

Lemma 3: The leading term of $\det(<,>^\eta)$ equals, up to a nonzero constant,
$$\prod_{\alpha\in\Delta^+}\prod_{n\geq 1} h_\alpha^{P(\eta-n\alpha)}$$

Exercise: Prove this lemma.

We shall next deduce a factorizable nature of $\det(<,>^\eta)$. First:

Def 3: $\beta\in Q_+\setminus\{0\}$ is called a "quasiroot" if $\exists\,\alpha\in\Delta^+$ s.t. β is multiple of α.

Lemma 4: $\det(<,>^\eta)$ equals, up to a nonzero factor, a product of linear factors $h_\beta+\rho(h_\beta)-\frac{1}{2}(\beta,\beta)$, β - quasiroot

- If M_λ is not irreducible, then $\exists\,\beta\in Q_+\setminus\{0\}$ and an embedding $M_{\lambda-\beta}\hookrightarrow M_\lambda$. But then the Casimir operator must act by the same constants, so that $(\lambda,\lambda+2\rho)=(\lambda-\beta,\lambda-\beta+2\rho)\Rightarrow(\lambda+\rho,\beta)=\frac{(\beta,\beta)}{2}\Rightarrow(\lambda+\rho)(h_\beta)=\frac{(\beta,\beta)}{2}$.

So: $\det(<,>^\eta)\in\mathbb{C}[\mathfrak{h}^*]$ has a zero set in the union of hyperplanes given by $\{(\lambda+\rho)(h_\beta)=\frac{1}{2}(\beta,\beta)\}_{\beta\in Q_+\setminus\{0\}}$.

Exercise: Deduce that then $\det(<,>^\eta)$ is a product of $h_\beta+\rho(h_\beta)-\frac{(\beta,\beta)}{2}$ (with $\beta\in Q_+\setminus\{0\}$), up to a nonzero constant.

However, using Lemma 3, we see that all β that do appear in this factorization must be quasiroots.